Lebesgue integration. Detailed proofs to be formalized in Coq

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Lebesgue integration.
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Abstract: To obtain the highest confidence on the correction of numerical simulation programs implementing the finite element method, one has to formalize the mathematical notions and results that allow to establish the soundness of the method. Sobolev spaces are the mathematical framework in which most weak formulations of partial derivative equations are stated, and where solutions are sought. These functional spaces are built on integration and measure theory. Hence, this chapter in functional analysis is a mandatory theoretical cornerstone for the definition of the finite element method. The purpose of this document is to provide the formal proof community with very detailed pen-and-paper proofs of the main results from integration and measure theory.

Key-words: measure theory, Lebesgue integration, detailed mathematical proof, formal proof in real analysis

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Intégrale de Lebesgue.
Preuves détaillées en vue d'une formalisation en Coq

Résumé : Pour obtenir la plus grande confiance en la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il faut formaliser les notions et résultats mathématiques qui permettent d'établir la justesse de la méthode. Les espaces de Sobolev sont le cadre mathématique dans lequel la plupart des formulations faibles pour résoudre les équations aux dérivées partielles sont posées, et où les solutions sont recherchées. La construction de ces espaces fonctionnels repose sur le calcul intégral et la théorie de la mesure. Ce chapitre de l'analyse fonctionnelle est donc un fondement théorique nécessaire à la définition de la méthode des éléments finis. L'objectif de ce document est de fournir à la communauté des chercheurs en preuve formelle des preuves papiers très détaillées des principaux résultats du calcul intégral et de la théorie de la mesure.

Mots-clés : théorie de la mesure, intégrale de Lebesgue, preuve mathématique détaillée, preuve formelle en analyse réelle

Foreword

This document is intended to evolve over time. Last version is release 1.1 (i.e. version 2). It is available at `https://hal.inria.fr/hal-03105815/`.

Version 2 (release 1.1, 2021/04/01) is a minor revision.
Main changes are:

- addition of this foreword;
- reordering and sectioning:
  sections have become chapters and gathered into parts to ease the reading;
  all "complements" are gathered into a single chapter;
  Section 7.2 about algebraic structures is moved right after Section 7.1 about set theory;
  Chapter 8 on subset systems is separated from Chapter 9 on measurability;
- some color modifications to increase legibility after grayscale printing;
- new contents in the introduction (Chapter 1), and minor corrections in Chapters 2–5 (including a bunch of hyperlinks in Chapters 4 and 5);
- fix proofs of uniqueness in Lemma 668 (statement changed too) and Theorem 724;
- fix statement and proof of Theorem 846 and Lemma 847, with new Definition 844;
- fix statement of Theorem 817;
- alternate proof of additivity for the integral of nonnegative simple functions in Lemma 774 based on the new *disjoint* representation of simple functions of Lemma 754;
  this involves new Definition 207, new Lemmas 734, 735, 756, 757, 764, 766, 768, and 772, new Remarks 208 750, 753, 755, 758, and 773, addition of uniqueness in Lemmas 752 and 765, thus impacting proofs of Lemmas 742, 744, 771, 779, 784, and 814;
- factor the proof on *layers* into new Lemmas 446, 480, and 711, thus impacting proofs of Lemmas 617, 715, and 719;
- new Lemma 644 and Remark 645 about reasoning with properties satisfied almost everywhere, now used in the proofs of Lemmas 651, 652, 653, 654, 659, and 664 (statement changed too in the latter);
- new Lemmas 376, 377, 600, and 601 shortening proofs of Lemmas 827, and 828;
- additional Remarks 214, 430, 473, 507, 512, 689, 723, 730, 795, 816, 845, 850, 896, 898, and modification of Remarks 407, 436, 504, and 674;
- Lemmas 552, and 553 are moved from Section 13.4 to Section 9.4;
- and possibly other slight modifications to improve the proof contents and their legibility.

Version 1 (release 1.0, 2021/01/14) is the first release.
It covers:

- the main basic results in measure theory, including the construction of Lebesgue measure in $\mathbb{R}$ via Carathéodory's extension scheme;
- integration of nonnegative measurable functions, including the Beppo Levi (monotone convergence) theorem and Fatou's lemma;
- the integral over a product space, including the Tonelli theorem;
- integration of measurable functions with possibly changing sign, including the seminormed vector space $\mathcal{L}^1$, and Lebesgue's dominated convergence theorem.

Contents

Part I

Overview

Chapter 1

Introduction

1.1 Formal proof

A formal proof is conducted in a logical framework that provides dedicated computer programs to mechanically check the validity of the proof, the so-called formal proof assistants. Such formal proofs may concern known mathematical theorems, but also properties of some piece of other computer programs, e.g. see [34], and [6, Glossary p. 343]. This field of computer science is extremely popular as it allows to certify with no doubt the behavior of critical programs.

Interactive theorem provers are now known to be able to tackle real analysis. For instance, in the field of Ordinary Differential Equations (ODEs) with Isabelle/HOL [37, 36, 38], and Coq [43], or in the field of Partial Differential Equations (PDEs), again with Isabelle/HOL [1], and Coq [5, 6]. In the latter example, the salient aspect is that the round-off error due to the use of IEEE-754 floating-point arithmetic can also be fully taken into account. But the price to pay is that every details of the proofs have to be dealt with, and thus the availability of very detailed pen-and-paper proofs is a major asset.

1.2 Objective

Our long term purpose is to formally prove programs implementing the Finite Element Method (FEM). The FEM is widely used to solve a broad class of PDEs, mainly because it has a sound mathematical foundation, e.g. see [50, 16, 45, 10, 24]. The Lax–Milgram theorem, one of the key ingredients to establish the FEM, was already addressed in [17] for a detailed pen-and-paper proof, and in [7] for a formal proof in Coq. The present document is a further contribution toward our ultimate goal.

The Lax–Milgram theorem claims existence and uniqueness of the solution to the weak formulation of a PDE problem, such as the Poisson problem, and its discrete approximation; it is stated on a Hilbert space, i.e. a complete inner product space. The simplest Hilbert functional spaces relevant for the resolution of PDEs are the spaces L^2 and H^1. More generally, when stronger results such as the Banach–Nečas–Babuška theorem are involved, one may be interested in the Sobolev spaces $W^{m,p}$ that are Banach spaces, i.e. complete normed vector spaces.

Let $d \in \{1, 2, 3\}$ be the spatial dimension. Let Ω be an open subset of $\mathbb{R}^d$. We consider real-valued functions defined almost everywhere over Ω. For $0 < p \leqslant \infty$, the Lebesgue space $L^p(\Omega)$ is the space of measurable functions for which the p-th power of the absolute value has a finite integral over Ω. For any natural number m, the Sobolev space $W^{m,p}(\Omega)$ is the vector subspace of $L^p(\Omega)$ of functions for which all weak derivatives up to order m also belong to $L^p(\Omega)$. Thus, Lebesgue spaces correspond to the case $m = 0$, $L^p(\Omega) = W^{0,p}(\Omega)$. And the Hilbert spaces correspond to the case $p = 2$, $H^m(\Omega) \stackrel{\text{def.}}{=} W^{m,2}(\Omega)$.

1.3 Integration theories

There is a huge variety of concepts of integral and integrability in the literature, e.g. see [15, 12], and one may wonder which one to use. Some are overridden by others, some are equivalent, and some have been developed for specific situations, such as vector-valued functions or functions defined on an infinite-dimensional domain. But a few of them have gained popularity, be it for their appropriateness for teaching, or their general-purpose nature: namely Riemann, Lebesgue, and Henstock–Kurzweil integrations.

Lebesgue integral (with Lebesgue measure) is the traditional framework in which the Sobolev functional spaces are defined. Indeed, Riemann integral is disqualified because of its poor results on limit and integral exchange making completeness unreachable, and so is Henstock–Kurzweil integral, because of its not so obvious extension to the multidimensional case and construction of a complete normed vector space of HK-integrable functions [29, 44].

1.4 Contents

The present version of this document covers all material up to the first properties of the seminormed vector space $\mathcal{L}^1$ of integrable functions (before taking the quotient to obtain the normed vector space L^1). Among the rich literature on Lebesgue integral theory, it was mostly derived from the textbooks [42, 28, 47].

It includes results on the general concepts of measurability, measure and negligibility, and integration of nonnegative measurable functions culminating with the Beppo Levi (monotone convergence) theorem allowing to exchange limit and integral for nondecreasing sequences of measurable functions, and Fatou's lemma that only gives an upper bound when the sequence is not monotone. The formalization in Coq of all these aspects is presented in [8].

In addition, this document also covers the building of the Lebesgue measure in $\mathbb{R}$ using Carathéodory's extension scheme, the integral over a product space including the Tonelli theorem (for nonnegative measurable functions, whereas the Fubini theorem deals with integrable functions), and the integral for measurable functions with possibly changing sign, including Lebesgue's dominated convergence theorem.

It is planned to add more results in a forthcoming version.

1.5 Teaching

This document is not primarily meant for teaching usage. The objective was to be as comprehensive as possible in the proofs. This led to very detailed demonstrations, and to a compact style of writing that is not common, and may seem daunting to the uninformed reader.

However, the authors tried to give some insights on the integration theory and on the proofs and theorems in the next introductory sections. They also strove to give some indications in the proofs when they felt it necessary. They believe that this document could be useful for interested teachers, and dedicated students.

1.6 Disclaimer

Note that the manuscript itself is not formally proved (and will never be). Indeed, LaTeX compilers are not formal proof tools.

Moreover, formalization is not just straightforward translation of mathematical texts and formulas. Some design choices have to be made and proof paths may differ, mainly to favor usability of Coq theorems and ease formal developments. Thus, there exist various differences between the mathematical setting presented here, and the formal setting developed in Coq [8]. For instance, the Coq definition of measurability of subsets (i.e. of σ-algebra) takes the generators as parameter, and not only the basic axioms. The ubiquitous use of *total* functions (defined for *all* values of their arguments) may be surprising at first: e.g. the addition in $\overline{\mathbb{R}}$ ("$\infty - \infty$" is defined and takes the value 0), or the measure (one can take the measure of any subset, even nonmeasurable ones). Of course, just as in mathematical statements, unwanted behaviors are somehow prevented, for instance by adding an hypothesis stating the legality of the addition or the measurability of the subset (see [8] for details).

Hence, despite the care taken in its writing, this document might still be prone to errors or holes in the demonstrations. There could also exist simpler paths in the proofs. Please, feel free to inform the authors of any such issue, and to share any comments or suggestions...

1.7 Organization

Part I of this document is organized as follows. The notations are first collected in Chapter 2. Then, the literature on the subject is briefly reviewed in Chapter 3. Chapter 4 gathers some proof techniques. The chosen proof paths of the main results are then sketched in Chapter 5.

Part II (Chapters 6 to 14) is the core of this document. In this part, the definitions are presented, and the lemmas and theorems are stated with their detailed proofs. Its organization is described at the end of the introductory Chapter 6.

Chapter 15 concludes and gives some perspectives.

Finally, an appendix gathers the list of statements in Chapter A, and explicit dependencies (both ways) in Chapters B and C. The appendix is not intended for printing!

Chapter 2

Notations

In this chapter (as in most of the document), we use the following conventions:

- capital letters X and Y denote "surrounding" sets, typically the domain and codomain of functions, A and B are subsets of X, and I denotes a set of indices;
- calligraphic letter $\mathcal{R}$ denotes a binary relation on X (e.g. equality or inequality);
- the capital letter P denotes a predicate, i.e. a function taking Boolean values;
- small letters x, y and i denote elements of the set using the matching capital letter;
- small letters a and b are (extended) real numbers (such that $a \leqslant b$), and n and p are (extended) natural numbers (such that $n \leqslant p$);
- small letter f denotes a function, e.g. from set X to set Y;
- the capital Greek letter Σ denotes a σ-algebra on X, and the small Greek letter μ denotes a measure on the measurable space (X, Σ).

The following notations and conventions are used throughout this document.

- Logic:
  - Using a compound (tuple of elements of X, or subset of X) in an expression at a location where only a single element makes sense is a shorthand for the same expression expanded for all elements of the compound; for instance, "$\forall x, x' \in X$" means "$\forall x \in X, \forall x' \in X$", "$\forall (x_i)_{i\in I} \in X$" means "$\forall i \in I,\ x_i \in X$", and "$x, x'\mathcal{R}x''$" means $x\mathcal{R}x''$ and $x'\mathcal{R}x''$;
  - $x_0\mathcal{R}_1x_1 \ldots \mathcal{R}_mx_m$ is a shorthand for $x_0\mathcal{R}_1x_1 \wedge x_1\mathcal{R}_2x_2 \wedge \ldots \wedge x_{m-1}\mathcal{R}_mx_m$;
  - "iff" is a shorthand for "if and only if".
- Set theory:
  - $\mathcal{P}(X)$ denotes the power set of X, i.e. the set of its subsets;
  - subsets are denoted using $\subset$, and proper subsets by $\subsetneq$;
  - n-ary set operations, such as intersection and union, have precedence over binary set operations (no need for big parentheses); for instance, $\bigcup_{i\in I} A_i \cap B$ means $\left(\bigcup_{i\in I} A_i\right) \cap B$;
  - superscript ^{c} denotes the absolute complement;
  - $\setminus$ denotes the set difference (or relative complement): $A \setminus B \overset{\text{def.}}{=} A \cap B^c$; when $B \subset A$, $A \setminus B$ is also called the *local complement of B (in A)*;
  - $\uplus$ denotes the disjoint union: $A \uplus B$ means $A \cup B$ with the assumption that $A \cap B = \varnothing$;
  - $\overline{\cap}$ is used to denote a set of traces of subsets, see Definition 216;
  - $\overline{\times}$ is used to denote a set of Cartesian products of subsets, see Definition 217;
  - $\mathrm{card}(X)$ is the cardinal of X, i.e. the number of its elements, with the convention $\mathrm{card}(X) \overset{\text{def.}}{=} \infty$ when X is infinite;
  - countable means either finite, or infinitely countable: X is countable iff there exists an injection from X to $\mathbb{N}$ iff there exists $I \subset \mathbb{N}$ and a bijection between I and X;
  - $\mathbb{1}_A$, or $\mathbb{1}_A^X$, denotes the indicator function of the subset A of X. It is the function from X to $\mathbb{R}$, or $\overline{\mathbb{R}}$, that takes the value 1 for all elements of A, and the value 0 elsewhere;
  - the set of functions from X to Y is either denoted Y^X, or through the type annotation "$X \to Y$". Both compact expressions "let $f \in Y^X$" and "let $f : X \to Y$" mean "let f be

a function from X to Y";
  - to avoid double parentheses/braces, the expressions involving inverse images of singletons are simplified: $f^{-1}(y)$ is a shorthand for $f^{-1}(\{y\})$;
  - $\{P(f)\}$ is a shorthand for $\{x \in X \mid P(f(x))\}$, or $f^{-1}(\{y \in Y \mid P(y)\})$; for instance, $\{f < a\}$ means either $f^{-1}[-\infty, a)$ or $f^{-1}(-\infty, a)$, see Lemma 570, and Lemma 578;
  - f^+ and f^- denote the nonnegative and nonpositive parts of a numerical function f, see Definition 399.

- Totally ordered set:
  - when the nonempty set X is totally ordered, intervals follow the notations used for real numbers; for instance, $(x_1, x_2]$ represents the subset $\{x \in X \mid x_1 < x \leqslant x_2\}$;
  - the square-and-round bracket notation is used for not specifying whether the bound is included or not: $(x_1, x_2⟧$ means either (x_1, x_2) or $(x_1, x_2]$;
  - the lower and greater bounds of X may be denoted $-\infty$ and $+\infty$ (as usual, the plus sign may be omitted); they may belong to X or not; $\overline{X}$ denotes $X \cup \{-\infty, \infty\}$.
  - rays, or half-lines, are denoted using the square-and-round notation: whether ∞ belongs to X or not, $(x, \infty⟧$ represents $\{x' \in X \mid x < x'\}$, i.e. either $(x, \infty]$ or (x, ∞), see Definition 241;
  - $\mathcal{I}_X^{o,p}$ denotes the set of open proper intervals of X, see Definition 241;
  - $\mathcal{R}_X^o$ denotes the set of open rays of X, see Definition 241.
  - $\mathcal{I}_X^o$ denotes the set of open intervals of X, see Definition 241.

- Numbers:
  - $\lfloor \cdot \rfloor$ denotes the floor function from $\mathbb{R}$ to $\mathbb{N}$: $\lfloor a \rfloor \leqslant a < \lfloor a \rfloor + 1$;
  - $⟦n..p⟧$ denotes the integer interval $⟦n, p⟧ \cap \mathbb{N}$ (with the natural variants using parentheses and/or brackets when exclusion or inclusion of bounds is specified);
  - $\ell(⟦a, b⟧)$ denotes the length of the interval $⟦a, b⟧$, see Definition 691.

- General topology:
  - $\mathcal{T}_X(G)$ denotes the topology generated by $G \subset \mathcal{P}(X)$, see Definition 251;

- Measure theory:
  - $\Pi_X(G)$ denotes the π-system generated by $G \subset \mathcal{P}(X)$, see Definition 432;
  - $\mathcal{A}_X(G)$ denotes the set algebra generated by $G \subset \mathcal{P}(X)$, see Definition 442;
  - $\mathcal{C}_X(G)$ denotes the monotone class generated by $G \subset \mathcal{P}(X)$, see Definition 450;
  - $\Lambda_X(G)$ denotes the λ-system generated by $G \subset \mathcal{P}(X)$, see Definition 463;
  - $\Sigma_X(G)$ denotes the σ-algebra generated by $G \subset \mathcal{P}(X)$, see Definition 482;
  - $\mathcal{B}(X)$ denotes the Borel σ-algebra generated by the open subsets, see Definition 517;
  - $\bigotimes_{i \in [1..m]} \Sigma_i$ denotes the tensor product of the σ-algebras $(\Sigma_i)_{i \in [1..m]}$, see Definition 541;
  - $\mathbf{N}$ denotes the set of negligible subsets, see Definition 631;
  - the annotation "μ a.e." specifies that the proposition is only considered almost everywhere, e.g. $\stackrel{\mu\,\text{a.e.}}{=}$, $\forall_{\mu\,\text{a.e.}}$, $(X \to Y)_{\mu\,\text{a.e.}}$, or $\mathcal{R}_{\mu\,\text{a.e.}}$, see Definition 641;
  - δ_Y denotes the counting measure associated with Y, δ_a denotes the Dirac measure at a, see Lemma 671 and Definition 675;
  - $\mathcal{L}$ denotes the Lebesgue σ-algebra on $\mathbb{R}$ and λ denotes the (Borel–)Lebesgue measure on $\mathbb{R}$, see Definitions 697 and 705, and Theorem 724;
  - $\mu_1 \otimes \mu_2$ denotes the tensor product of measures μ_1 and μ_2, see Definition 829, Definition 830, and Lemma 837;
  - $\lambda^{\otimes 2}$ denotes the Lebesgue measure on $\mathbb{R}^2$, see Lemma 839;

- Lebesgue integral:
  - $\mathcal{IF}$ denotes the set of measurable indicator functions, see Definition 732;
  - $\mathcal{SF}$ denotes the vector space of simple functions, see Definition 748;
  - $\mathcal{SF}_+$ denotes the set of nonnegative simple functions, see Definition 763;
  - $\mathcal{M}_+$ denotes the set of nonnegative measurable functions $X \to \overline{\mathbb{R}}_+$, see Definition 593;
  - $\mathcal{M}$ denotes the set of measurable functions $X \to \overline{\mathbb{R}}$, see Definition 575;
  - $\mathcal{M}_{\mathbb{R}}$ denotes the vector space of measurable functions $X \to \mathbb{R}$, see Definition 567;

- $\int f \, d\mu$ denotes the (Lebesgue) integral of f for the measure μ, whether the function belongs to $\mathcal{IF}$, $\mathcal{SF}_+$, $\mathcal{M}_+$, or $\mathcal{M}$, see Section 4.1, Definition 740, Lemmas 770 and 789, and Definition 858;
- $\int_A f \, d\mu$ denotes the integral of the restriction of f to A, see Lemmas 743, 783, 813, and 864;
- $\int_a^b f(x) \, d\mu(x)$ denotes the integral of f over the interval (a, b), see Lemma 867;
- $\int f(x) \, dx$ denotes the integral of f for the Lebesgue measure, see Definition 873;
- $(\mathcal{L}^1, N_1)$ denotes the seminormed vector space of integrable functions, see Lemma 874, and Definition 884;
- $\mathcal{I}$ denotes the integral operator for integrable functions, see Lemma 892.

Chapter 3

State of the art

After a brief survey of the wide variety of integrals, and of the means to build Lebesgue integral and Lebesgue measure, we review some works of a few authors, partly from the mathematical French school, that provide some details about results in measure theory, Lebesgue integration, and basics of functional analysis.

As usual, proofs provided in the literature are not comprehensive, and we have to cover a series of sources to collect all the details necessary for a formalization in a formal proof tool such as Coq. Usually, Lecture Notes in undergraduate mathematics are very helpful and we selected [31, 32, 33] among many other possible choices.

3.1 A zoology of integrals

The history of integral calculus dates back at least from the Greeks, with the method of exhaustion for the evaluation of areas, or volumes. Since then, mathematicians have constantly endeavored to develop new techniques. The objective is mainly to be able to integrate a wider class of functions. But also to establish more powerful results, such as convergence results, or the Fundamental Theorems of Calculus (expressing that derivation and integration are each other inverse operations). And possibly to fit specific contexts, e.g. driven by applications in physics, or for teaching purposes. For a wide panel, and pros and cons of different approaches, see for instance [18, 15, 12], in which more than a hundred named integrals are listed. But, as pointed out in [48], building a universal integral "will probably be unnecessarily complicated when restricted to a simple setting".

Considering the case of a numerical function defined on some interval, let us review some of the most popular ones.

3.1.1 Leibniz–Newton integral (late 17th century)

The Leibniz–Newton (LN-)integral is defined in a descriptive way as the difference of any primitive evaluated at the bounds of the integration interval. All continuous functions are LN-integrable.

This integral possesses interesting properties such as formulas for integration by parts or by substitution, and a uniform convergence theorem. But since the definition is not constructive, there is no regular process to build the primitive of a function, even if it is continuous. Moreover, other monotone or dominated convergence results must assume that the limit is LN-integrable. And it raises the question of integrability of functions with no primitives.

3.1.2 Riemann integral (early 19th century)

The Riemann (R-)integral [46] consists in cutting the area "under" the graph of the function into vertical rectangular strips by choosing a subdivision of the integration interval, and in increasing the number of strips by making the step of the subdivisions go down to 0. Equivalently, it might be more convenient to consider the Darboux integral [21], which checks that both upper and lower so-called Darboux sums have the same limit.

This simple approach has similar properties than the LN-integral, with for instance a uniform convergence result, and R-integrable functions are the piecewise continuous function which set of discontinuity points has null measure. But again, monotone convergence and dominated convergence theorems must assume R-integrability of the limit.

3.1.3 Lebesgue integral (early 20th century)

The key idea of the Lebesgue (L-) approach [40] is to consider subdivisions of the codomain of the function, i.e. cutting the area under the graph into horizontal pieces, which are no longer continuous strips when the function is not concave, and to define a measure for these pieces that generalizes the length of intervals.

This has the great advantage of providing powerful monotone and dominated convergence theorems, and to allow for an abstract setting in which one can handle functions defined on more general spaces than the Euclidean spaces $\mathbb{R}^n$. And this paves the road for probability theory. For instance, the Wiener measure used for the study of stochastic processes such as Brownian motion is defined on the infinite-dimensional Banach space of continuous functions from a compact interval to $\mathbb{R}^n$. Moreover, the concept of property satisfied *almost everywhere* opens the way to the L^p Lebesgue spaces as (complete) normed vector spaces (Banach spaces).

The main difficulty is the necessity to develop the measure theory to be able to associate a length to the "horizontal pieces". In general, this is not possible for all subsets of the domain of the function, and leads to the concepts of *measurable subset* and *measurable function*.

Going back to the geometrical interpretation of the area under the graph of the function, it is interesting to note that the opposition Riemann (subdivision of the domain) versus Lebesgue (subdivision of the codomain) reflects in the field of ordinary differential equations where the classical numerical schemes based on a discretization of the time domain are now opposed to the recent quantized state system solvers that are based on the quantization of the state variables, see [14, 25, 26]. Moreover, we may also cite [30] in the field of signal processing where the Lebesgue geometrical scheme is seen as a nonlinear sampling qualified of "noise-free quantization", that could have impact on electronic design.

3.1.4 Henstock–Kurzweil integral (mid-20th century)

The Henstock–Kurzweil (HK-)integral, or gauge integral [39, 35, 3] is a generalization of the R-integral for which the subdivision of the integration interval is no longer uniform, but is driven by the variations of the function to integrate through a so-called *gauge* function.

It is more powerful than the L-integral in the sense that a real-valued function is L-integrable if and only if the function and its absolute value are HK-integrable. Unlike the L-integral, this leads to formulations of the fundamental theorem of calculus without the need for the concept of improper integral. The monotone and dominated convergence theorems are also valid. Moreover, the HK-integral is usually considered more suited for teaching, for it is hardly more complex than the R-integral.

However, unlike the L-integral, the HK-integral cannot be easily extended to the Euclidean spaces $\mathbb{R}^n$, and the construction of a Banach space of HK-integrable function is far less obvious than that of the L^1 Lebesgue space [29, 44].

3.1.5 Daniell integral (early 20th century)

The Daniell (D-) approach [20, 27] is a general scheme that extends an *elementary* integral defined for *elementary* functions to a much wider class of functions by checking that upper and lower elementary integrals of elementary functions share the same limit, in a way that is similar to the Darboux approach. When applied to the integral of simple functions, this is somehow equivalent to steps 3 and 4 of the Lebesgue scheme described later in Section 4.1. But it is enough to consider the R-integral for the compactly supported continuous functions (a.k.a. the Cauchy integral) to get back the L-integral for the Lebesgue measure on the Euclidean spaces $\mathbb{R}^n$.

This Daniell approach to the L-integral has the great advantage not to need the concept of measure, which can be reconstructed afterwards by taking the integral of indicator functions of measurable subsets. Moreover, in the mid-20th century, this approach was shown in [4] to be compatible with constructive analysis.

In the present document, we choose the Lebesgue approach.

3.2 Lebesgue integral and Lebesgue measure

The Lebesgue integral was originally built using measure theory following the so-called Lebesgue scheme, described in Section 4.1. We have seen that it may also be built in a constructive way by applying the Daniell extension scheme to some elementary integral defined for elementary functions, such as the Riemann integral for compactly supported continuous functions. Another alternative using the same basic ingredients consists in the completion of the normed vector space of compactly supported continuous functions and the extension of the Riemann integral which is uniformly continuous [9, 22].

The Lebesgue measure on $\mathbb{R}$ is a generalization of the length of bounded intervals to a much wider class of subsets. This extension is actually unique and complete. It is defined on the Lebesgue σ-algebra which is generated by the Borel subsets and the negligible subsets. The restriction to the sole Borel subsets (the σ-algebra generated by the open subsets, or the closed subsets) is sometimes called the Borel(–Lebesgue) measure. In the very same way, the Lebesgue measure on the Euclidean spaces $\mathbb{R}^n$ is a generalization of the n-volume of rectangular boxes. It is the tensor product of the Lebesgue measure on $\mathbb{R}$.

There are three main techniques for the construction of the Lebesgue measure on $\mathbb{R}$. The most popular one for teaching is through Carathéodory's extension theorem [13, 23]. The process builds progressively the Lebesgue σ-algebra and the Lebesgue measure with their properties (see Section 4.2). A more abstract approach is based on the Riesz–Markov–Kakutani representation theorem, which associates any positive linear form on the space of compactly supported continuous functions with a unique measure, e.g. see [47]. The third way is not based on measure theory. It follows the Daniell approach to integration, which is also based on the Riemann integration of compactly supported continuous functions, and defines the Lebesgue measure of any measurable subset as the integral of its indicator function.

The construction of non-Lebesgue-measurable subsets requires the use the axiom of choice, as for instance the Vitali subset [49].

In the present document, we build Lebesgue integral using the Lebesgue scheme (see Section 4.1), and Lebesgue measure through Carathéodory's extension theorem (see Section 4.2).

3.3 Our main sources

The main ingredients of measure theory and Lebesgue integration are presented in an elementary manner in the textbook [42], which is very well suited for our purpose. However, some results such as the construction of the Lebesgue measure and the Tonelli theorem are admitted.

Details for the construction of the Lebesgue measure (through the Carathéodory approach), and for the proofs of the Fubini–Tonelli theorems can be found in [28]. This other textbook is quite comprehensive. Many results are presented as exercises with their solution.

We may also cite [47], [2], and [11]. The former for instance for a construction of the Lebesgue measure through the Riesz–Markov–Kakutani representation theorem. And the latter two for the L^p Lebesgue spaces and the $W^{m,p}$ Sobolev spaces.

Note that some original forms of statements have been developed for the present document, and even though the results are in general well known, the formulations are new to our knowledge. This includes:

- the individual treatment of constitutive properties (closedness under set operations) of subset systems in Section 8.1;
- the formalization of most concepts based on properties almost satisfied in Section 11.2 with abstract results such as Lemmas 659, 660, 661, 664, and 807 (the latter in Section 13.3);
- the concept of almost sum in Section 12.1 with Lemmas 682 and 683, and Lemmas 882 and 883 in Section 14.3;
- the concept of disjoint representation of simple functions in Section 13.2 with Lemmas 754, 756, and 757;
- the fully detailed proof of the technical Lemma 776 in Section 13.2 to obtain additivity of the integral of nonnegative simple functions in Lemma 778.

Chapter 4

Proof techniques

In the framework of integration theory, some proof techniques are used repeatedly. This chapter describes the most important ones: the "Lebesgue scheme", the "Carathéodory extension scheme", and the "Dynkin π–λ theorem / monotone class theorem scheme".

4.1 Lebesgue scheme

Consider a set X equipped with a σ-algebra Σ of measurable subsets, and a measure μ that associates a "length" (in $\overline{\mathbb{R}}_+$) to any measurable subset. The set of extended real numbers $\overline{\mathbb{R}}$ is equipped with the Borel σ-algebra $\mathcal{B}(\overline{\mathbb{R}})$ that is generated by all open subsets. The purpose is to build, or prove properties of, the integral of "measurable" functions from X to $\overline{\mathbb{R}}$, i.e. for which the inverse image of measurable subsets (of $\overline{\mathbb{R}}$) are measurable subsets (of X).

The Lebesgue scheme consists in establishing facts about the integral by working successively inside three embedded functional subsets: the set $\mathcal{IF}$ of indicator functions of measurable subsets, the vector space $\mathcal{SF}$ of simple functions (the linear span of $\mathcal{IF}$), and the set $\mathcal{M}$ of measurable functions.

The four steps of the scheme are the following:

1. establish the fact in the case of indicator functions (in $\mathcal{IF}$) for which the integral is simply the measure of the support;

2. then, generalize the fact to the case of nonnegative simple functions (in $\mathcal{SF}_+$) by nonnegative (finite) linear combination of indicator functions;

3. then again, generalize the fact to the case of nonnegative measurable functions (in $\mathcal{M}_+$) by taking the supremum for all lower nonnegative simple functions;

4. finally, generalize the fact to the case of all measurable functions (in $\mathcal{M}$) by taking the difference between the expressions involving nonnegative and nonpositive parts (integrable functions are those for which this difference is well-defined).

Actually, a fifth step can be added to generalize the fact to the case of numeric functions taking their values in $\mathbb{R}^n$, $\mathbb{C}$, or $\mathbb{C}^n$ by considering separately all components.

In the present document, the steps of Lebesgue scheme are mainly used to build the Lebesgue integral (steps 1 to 4, see Chapter 13 and Section 14.1), but also to establish some properties such as the Beppo Levi (monotone convergence) theorem (step 3, see Section 5.5 and Theorem 796) and the Tonelli theorem (steps 1 to 3, see Section 5.3 and Theorem 846).

4.2 Carathéodory's extension scheme

The Carathéodory extension scheme consists in applying the eponymous theorem (see Section 5.6) to extend a pre-measure (i.e. defined on a ring of subsets, and not on a σ-algebra) into a complete measure on a σ-algebra containing the initial ring of subsets.

Let X be the ambient set, μ be the pre-measure and $\mathcal{R}$ be the ring of subsets (of X). The two steps of the scheme are the following:

1. extend the pre-measure to any subset of X by taking the infimum of the sum of pre-measures of all coverings with elements of the ring $\mathcal{R}$ (and the value ∞ when there is no such covering):

$$\mu^\star(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n\in\mathbb{N}} \mu(A_n) \;\middle|\; (A_n)_{n\in\mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n\in\mathbb{N}} A_n \right\};$$

   the theorem ensures that $\mu^\star$ is an outer measure on X, and that it is an extension of the pre-measure μ;

2. define the set of Carathéodory-measurable subsets as

$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \mid \forall A \subset X,\ \mu^\star(A) = \mu^\star(A \cap E) + \mu^\star(A \setminus E)\};$$

   the theorem ensures that Σ is a σ-algebra on X containing the ring $\mathcal{R}$, and that the restriction ${\mu^\star}_{|\Sigma}$ is a complete measure on (X, Σ).

The Carathéodory extension scheme is a popular tool to build measures. The prominent example is the Lebesgue measure on $\mathbb{R}$ that extends the length of open intervals (see Section 12.2). But for instance, the extension scheme may also be used to build the Lebesgue-Stieltjes measure associated with its cumulative distribution function, and Loeb measures in a nonstandard analysis framework, e.g. see [41, 19].

4.3 Dynkin π–λ theorem / monotone class theorem schemes

A π-system, a set algebra, a monotone class, and a λ-system are all different kinds of subset systems (i.e. subsets of the power set), just like a σ-algebra, but with less basic properties.

Roughly speaking, being a λ-system (resp. a monotone class) is what is missing to a π-system (resp. an algebra of subsets) to be a σ-algebra. To prove that some property P holds on some σ-algebra Σ generated by some subset system G, the idea is to proceed in two steps. First, show that the property P holds on the simpler kind of subset system (π-system, or set algebra) generated by G, and then establish that the subset of Σ where P holds is of the less simple kind of subset system (λ-system, or monotone class). Indeed, the Dynkin π–λ theorem states that the λ-system generated by a π-system is equal to the σ-algebra generated by the same π-system, and similarly, the monotone class theorem states that the monotone class generated by a set algebra is equal to the σ-algebra generated by the same set algebra (see Section 5.7). Thus, P is established on the whole σ-algebra.

The five steps of the scheme of the Dynkin π–λ theorem, and of the monotone class theorem are the following:

1. first define $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$; obviously, we have $\mathcal{S} \subset \Sigma$;

2. then, show that $G \subset \mathcal{S}$;

3. then, show that the π-system (resp. set algebra) $\mathcal{U}^2_X(G)$ is a subset of $\mathcal{S}$;

4. then, show that $\mathcal{S}$ is a λ-system (resp. monotone class);

5. and finally, obtain the other inclusion $\Sigma \subset \mathcal{S}$, because from monotonicity of subset system generation, and from the Dynkin π–λ theorem (resp. monotone class theorem), i.e. the equality in the middle, we have

$$\Sigma = \Sigma_X(G) \subset \Sigma_X(\mathcal{U}_X^2(G)) = \mathcal{U}_X^1(\mathcal{U}_X^2(G)) \subset \mathcal{U}_X^1(\mathcal{S}) = \mathcal{S}$$

where $\mathcal{U}_X^1 = \Lambda_X$ (resp. $\mathcal{C}_X$), i.e. the generated λ-system (resp. monotone class), and $\mathcal{U}_X^2 = \Pi_X$ (resp. $\mathcal{A}_X$), i.e. the generated π-system (resp. set algebra).

In the present document, the fifth step is embodied by Lemma 510 for the Dynkin π–λ theorem, and by Lemma 515 for the monotone class theorem. The Dynkin π–λ theorem scheme is used once when extending the equality of two measures on a generator π-system to the whole σ-algebra in Lemma 668, which is then used to establish uniqueness of the Lebesgue measure on $\mathbb{R}$ in Theorem 724 (see also Section 5.6). The monotone class theorem scheme is used twice when building the tensor product measure in the context of the product of finite measure spaces: to establish first the measurability of the measure of sections in Lemma 827, and then the uniqueness of the tensor product measure in Lemma 835 (both in the finite case).

Chapter 5

Statements and sketches of the proofs

This chapter gathers the sketches of the proofs of the main results that are detailed in Part II. Namely: Lebesgue's dominated convergence theorem and its extended version, the Tonelli theorem, Fatou's lemma, the Beppo Levi (monotone convergence) theorem, Carathéodory's extension theorem, the Dynkin π–λ theorem, and the monotone class theorem.

5.1 Sketch of the proof of Lebesgue's extended dominated convergence theorem

Lebesgue's extended dominated convergence theorem.
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}}, f, g \in \mathcal{M}$. Assume that the sequence is μ-almost everywhere pointwise convergent towards f. Assume that g is μ-integrable, and that for all $n \in \mathbb{N}$, we have $|f_n| \overset{\mu\text{ a.e.}}{\leqslant} g$. Then, for all $n \in \mathbb{N}$, f_n is μ-integrable, f is μ-integrable, and we have in $\mathbb{R}$

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu. \tag{5.1}$$

See Theorem 899. The proof of Lebesgue's extended dominated convergence theorem goes as follows (this proof uses the arithmetic of $\overline{\mathbb{R}}$, but with functions that are almost everywhere finite):

- $D \overset{\text{def.}}{=} \{f = \liminf_{n\to\infty} f_n\} \cap \{f = \limsup_{n\to\infty} f_n\} \cap \bigcap_{n\in\mathbb{N}} \{|f_n| \leqslant g\} \cap g^{-1}(\mathbb{R}_+) \subset X$ is first shown to be measurable with $\mu(D^c) = 0$; thus $f_n \mathbb{1}_D \overset{\mu\text{ a.e.}}{=} f_n$, $f\mathbb{1}_D \overset{\mu\text{ a.e.}}{=} f$, and $g\mathbb{1}_D \overset{\mu\text{ a.e.}}{=} g$; moreover, $g\mathbb{1}_D$ belongs to $\mathcal{L}^1$;
- then, since $\lim_{n\to\infty} f_n\mathbb{1}_D = f\mathbb{1}_D$ and $|f_n\mathbb{1}_D| \leqslant g\mathbb{1}_D$, Lebesgue's dominated convergence theorem (see Section 5.2) provides $f_n\mathbb{1}_D, f\mathbb{1}_D \in \mathcal{L}^1$, and the equality

$$\int f\mathbb{1}_D\,d\mu = \lim_{n\to\infty} \int f_n\mathbb{1}_D\,d\mu;$$

- finally, the integrability of f_n and f, and identity (5.1) follow from the compatibility of the integral in $\mathcal{M}$ with almost equality.

For instance, Lebesgue's extended dominated convergence theorem may be used to prove the Leibniz integral rule (differentiation under the integral sign), e.g. for the study of integrals function of their upper bound.

5.2 Sketch of the proof of Lebesgue's dominated convergence theorem

Lebesgue's dominated convergence theorem.
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}$. Assume that the sequence is pointwise convergent towards f. Let $g \in \mathcal{L}^1$. Assume that for all $n \in \mathbb{N}$, $|f_n| \leqslant g$. Then, for all $n \in \mathbb{N}$, $f_n \in \mathcal{L}^1$, $f \in \mathcal{L}^1$, the sequence is convergent towards f in $\mathcal{L}^1$, and we have in $\mathbb{R}$

$$\int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu. \tag{5.2}$$

See Theorem 897. The proof of Lebesgue's dominated convergence theorem goes as follows (this proof uses the arithmetic of $\mathbb{R}$):

- monotonicity of integral in $\mathcal{M}_+$ provides $f_n, f \in \mathcal{L}^1$;
- the sequence $g_n \stackrel{\text{def.}}{=} |f_n - f|$ is shown to be in $\mathcal{L}^1 \cap \mathcal{M}_+$ with limit 0;
- the sequence $2g - g_n$ is also shown to be in $\mathcal{L}^1 \cap \mathcal{M}_+$, with limit inferior $2g$;
- then, Fatou's lemma (see Section 5.4) and linearity of the integral in $\mathcal{L}^1$ provide the inequality
$$2 \int g \, d\mu \leqslant 2 \int g \, d\mu - \limsup_{n\to\infty} \int g_n \, d\mu;$$
thus, a nonnegativeness argument provides the nullity of the limit of the integral of g_n's, i.e. the convergence of the f_n's towards f in $\mathcal{L}^1$;
- finally, identity (5.2) follows from nondecreasingness of the integral in $\mathcal{L}^1$, the squeeze theorem, and linearity of the limit.

Lebesgue's dominated convergence theorem is used in the present document to prove Lebesgue's extended dominated convergence theorem (see Section 5.1 and Theorem 899).

Lebesgue's dominated convergence theorem admits several variants: with lighter assumptions, or set in L^p for $p \in [1, \infty)$. For instance, it may be used to establish the Fourier inversion formula, or to study the Gibbs phenomenon.

5.3 Sketch of the proof of the Tonelli theorem

Tonelli theorem.
*Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$.
Let $i \in \{1, 2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let ψ be the permutation $((x_i, x_j) \mapsto (x_1, x_2))$.
For all $x_i \in X_i$, let $f_{x_i} \stackrel{\text{def.}}{=} (x_j \mapsto f \circ \psi(x_i, x_j))$. Let $I_{f,i} \stackrel{\text{def.}}{=} (x_i \mapsto \int f_{x_i} \, d\mu_j)$.
Then, for all $x_i \in X_i$, $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$, $I_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i)$, and we have in $\overline{\mathbb{R}}_+$*

$$\int f \, d(\mu_1 \otimes \mu_2) = \int I_{f,i} \, d\mu_i. \tag{5.3}$$

See Theorem 846. The proof of the Tonelli theorem goes as follows (this proof uses the arithmetic of $\overline{\mathbb{R}}_+$ and the concepts of set algebra and monotone class; it follows steps 1 to 3 of the Lebesgue scheme, see Section 4.1):

- the result is first established for indicator functions from properties of the measure (in particular the building of the tensor product measure that uses continuity of measures from below and from above);

- then, the result is extended to nonnegative simple functions by taking (positive) linear combination of indicator functions and applying (positive) linearity of the integral in $\mathcal{M}_+$;
- and finally, the result is extended to nonnegative measurable functions by taking the supremum of adapted sequences and applying the Beppo Levi (monotone convergence) theorem (see Section 5.5).

The Tonelli theorem is used in the present document to establish identities: for an integral over a subset in Lemma 847, and for an integral of a tensor product function in Lemma 848.

5.4 Sketch of the proof of Fatou's lemma

Fatou's lemma. *Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Then, $\liminf_{n\to\infty} f_n \in \mathcal{M}_+$, and we have in $\overline{\mathbb{R}}_+$*

$$\int \liminf_{n\to\infty} f_n \, d\mu \leqslant \liminf_{n\to\infty} \int f_n \, d\mu. \tag{5.4}$$

See Theorem 817. The proof of Fatou's lemma goes as follows (this proof uses the arithmetic of $\overline{\mathbb{R}}_+$):

- the sequence $(\inf_{p\in\mathbb{N}} f_{n+p})_{n\in\mathbb{N}}$ is first shown to be pointwise nondecreasing in $\mathcal{M}_+$;
- then, the Beppo Levi (monotone convergence) theorem (see Section 5.5) provides the equality
$$\int \liminf_{n\in\mathbb{N}} f_n \, d\mu = \lim_{n\in\mathbb{N}} \int \inf_{p\in\mathbb{N}} f_{n+p} \, d\mu;$$
- and monotonicity of integral, and infimum and limit inferior definitions provide the inequality
$$\lim_{n\in\mathbb{N}} \int \inf_{p\in\mathbb{N}} f_{n+p} \, d\mu \leqslant \liminf_{n\in\mathbb{N}} \int f_n \, d\mu.$$

Fatou's lemma is used in the present document to prove properties of the integral in $\mathcal{M}_+$: identity for pointwise convergent sequences in Lemma 818, and Lebesgue's dominated convergence theorem (see Section 5.2 and Theorem 897).

The Beppo Levi (monotone convergence) theorem and Fatou's lemma can be established independently of one another, or each one can be obtained as a consequence of the other. Of course, those independent proofs do share a common ingredient to somehow obtain an upper bound for the integral of the limit by the limit of the integrals. We chose to establish the Beppo Levi (monotone convergence) theorem first.

5.5 Sketch of the proof of the Beppo Levi (monotone convergence) theorem

Beppo Levi (monotone convergence) theorem.
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Assume that the sequence is pointwise nondecreasing. Then, $\lim_{n\to\infty} f_n \in \mathcal{M}_+$, and we have in $\overline{\mathbb{R}}_+$

$$\int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu. \tag{5.5}$$

See Theorem 796. The proof of the Beppo Levi (monotone convergence) theorem goes as follows (this proof uses the arithmetic of $\overline{\mathbb{R}}_+$, it follows step 3 of the Lebesgue scheme by taking the "supremum" of quantities involving simple functions, see Section 4.1):

- let $f \stackrel{\text{def.}}{=} \lim_{n\to\infty} f_n$; monotonicity and completeness arguments provide that f is also $\sup_{n\to\infty} f_n$, that it belongs to $\mathcal{M}_+$, and the "easy" inequality
$$\lim_{n\to\infty} \int f_n \, d\mu \leqslant \int f \, d\mu;$$
- let $\varphi \in \mathcal{SF}_+$ such that $\varphi \leqslant f$; using the nondecreasing sequence of measurable subsets $(\{a\varphi \leqslant f_n\})_{n\in\mathbb{N}}$ for $a \in (0,1)$, continuity from below of μ, linearity, monotonicity and continuity arguments, allows to show the inequality
$$\int \varphi \, d\mu \leqslant \lim_{n\to\infty} \int f_n \, d\mu;$$
- then, taking the supremum over φ's provides the other inequality
$$\int f \, d\mu \leqslant \lim_{n\to\infty} \int f_n \, d\mu.$$

The Beppo Levi (monotone convergence) theorem is used in the present document to prove properties of the integral in $\mathcal{M}_+$ (possibly through Lemma 800): homogeneity at ∞ in Lemma 797, additivity in Lemma 801, σ-additivity in Lemma 803, identity for the integral over a subset in Lemma 813, Fatou's lemma (see Section 5.4 and Theorem 817), identity for the integral with the counting measure in Lemma 819, and the Tonelli theorem (see Section 5.3 and Theorem 846).

5.6 Sketch of the proof of Carathéodory's extension theorem

Carathéodory's extension theorem.
Let X be a set. Let $\mathcal{R}$ be a ring of subsets on X (i.e. closed under complement and union). Let μ be a pre-measure defined on $\mathcal{R}$ (i.e. null on the empty set, and σ-additive). Then, μ can be extended into a complete measure on a σ-algebra containing $\mathcal{R}$ (i.e. for which all negligible subsets are measurable, and of measure 0). Moreover, this extension is unique when μ is σ-finite.

See Theorem 724 (in the specific case of $X \stackrel{\text{def.}}{=} \mathbb{R}$ and $\mathcal{R}$ is the ring generated by the intervals). The proof of Carathéodory's extension theorem goes as follows (this proof uses the arithmetic of $\overline{\mathbb{R}}_+$ and the concepts of set algebra and monotone class; the existence part may also be known as the Carathéodory extension scheme, see Section 4.2):

- the function $\mu^\star$ defined on $\mathcal{P}(X)$ by (with the convention $\inf \varnothing = \infty$)
$$\forall A \subset X, \quad \mu^\star(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n\in\mathbb{N}} \mu(A_n) \;\middle|\; (A_n)_{n\in\mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n\in\mathbb{N}} A_n \right\}$$
is first shown to be an outer measure on X (i.e. null on the empty set, nonnegative, monotone, and σ-subadditive) whose restriction to $\mathcal{R}$ is the pre-measure μ;
- then, the set of subsets
$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \,|\, \forall A \subset X,\ \mu^\star(A) = \mu^\star(A \cap E) + \mu^\star(A \setminus E)\}$$
is shown to be a σ-algebra on X containing the ring $\mathcal{R}$, while the restriction ${\mu^\star}_{|\Sigma}$ is shown to be a complete measure on (X, Σ) (closedness of Σ under countable union comes from additivity of $\mu^\star$ on Σ);
- moreover, when μ is σ-finite (with $X = \biguplus_{n\in\mathbb{N}} X_n$ and $\mu(X_n) < \infty$), two Carathéodory extensions μ_1 and μ_2 are first restricted to X_n, then the set $\{\mu_1(A \cap X_n) = \mu_2(A \cap X_n)\}$ is shown to be equal to Σ, and finally uniqueness follows from σ-additivity of μ_1 and μ_2.

In the present document, Carathéodory's extension theorem is stated (and proved) in the specific case of the building of the Lebesgue measure on $\mathbb{R}$ (see Section 12.2).

5.7 Sketch of the proof of the Dynkin π–λ theorem / monotone class theorem

Both theorems take advantage of the complementarity of being a λ-system and being a π-system (for the Dynkin π–λ theorem), or being a monotone class and being a set algebra (for the monotone class theorem). Using the notation of Section 4.3 where $\mathcal{U}_X^1$ represents the generated λ-system Λ_X (resp. the generated monotone class $\mathcal{C}_X$), and $\mathcal{U}_X^2$ represents the generated π-system Π_X (resp. the generated set algebra $\mathcal{A}_X$), both theorems share the following abstract form:

Dynkin π–λ theorem / monotone class theorem.
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{U}_X^2(G) = G$. Then, $\mathcal{U}_X^1(G) = \Sigma_X(G)$.

See Theorems 508 and 513. Their common proof goes as follows (these proofs use the concepts of π-system and λ-system for the Dynkin π–λ theorem, and of set algebra and monotone class for the monotone class theorem):

- first, prove that for all G', $\mathcal{U}_X^1(G') = G'$ and $\mathcal{U}_X^2(G') = G'$ implies $\Sigma_X(G') = G'$;
- then, note that $\mathcal{U}_X^1(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$ (i.e. idempotent law for subset system generation);
- then, prove that $\mathcal{U}_X^2(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$ (i.e. the property of G is transmitted to $\mathcal{U}_X^1(G)$);
- then, apply the first result to $G' \stackrel{\text{def.}}{=} \mathcal{U}_X^1(G)$ and obtain $\Sigma_X(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$;
- and finally, prove that $\Sigma_X(\mathcal{U}_X^1(G)) = \Sigma_X(G)$.

In both cases, the most technical part is the third point.

The Dynkin π–λ theorem is used in the present document (through Lemma 510) to extend the equality of two measures on a generator π-system to the whole σ-algebra in Lemma 668, which is used to establish uniqueness of the Lebesgue measure on $\mathbb{R}$ in Theorem 724 (see also Section 5.6). The monotone class theorem is used in the present document (through Lemma 515) to prove measurability of the measure of sections (in the case of finite measure spaces) in Lemma 827, and uniqueness of the tensor product measure (also in the finite case) in Lemma 835. All these three proofs follow the Dynkin π–λ theorem / monotone class theorem scheme (see Section 4.3).

Part II

Detailed proofs

Chapter 6

Introduction

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

light gray is for remarks, light green for definitions,
light blue for lemmas, and light red for theorems.

Definitions and results have a number and a name. Inside the bodies of proofs, pertinent statements are referenced using both their number and name. When appropriate, some hints are given about the application, either to specify arguments, or to provide justification or consequences; they are underlined. Some useful definitions and results were already stated in [17], which was devoted to the detailed proof of the Lax–Milgram theorem. Those are numbered up to 206, and the statements in the present document are numbered starting from 207.

Furthermore, as in [17], the most basic results are supposed to be known and are not detailed further; they are displayed in **bold dark red**. These include:

- Logic: tautologies from propositional calculus.
- Set theory:
  - definition and properties of inclusion, intersection, (disjoint) union, complement, set difference, Cartesian product, power set, cardinality, and indicator function (that takes values 0 and 1), such as De Morgan's laws, monotonicity of intersection, distributivity of intersection over (disjoint) union and set difference, distributivity of the Cartesian product over union, compatibility of intersection with Cartesian product, σ-additivity of the cardinality (with ∞ absorbing element for addition in $\overline{\mathbb{N}}$);
  - definition and properties of equivalence and order binary relations;
  - definition and properties of function, composition of functions, inverse image (compatibility with set operations), injective and bijective functions, restriction and extension;
  - countability of finite Cartesian products of countable sets ($\mathbb{N}^2$, $\mathbb{Q}$, $\mathbb{Q} \times \mathbb{Q}_+^\star$).
- Algebraic structures: results from group theory.
- Topology: definition and properties of continuous functions.
- Real analysis:
  - properties of the ordered and valued field $\mathbb{R}$ such as the Archimedean property, density of rational numbers, completeness, sum of the first terms of a geometric series;
  - definition and properties of basic numeric analytic functions such as square root, power function, exponential function, natural logarithm function, and exponentiation;
  - properties of the ordered set $\overline{\mathbb{R}}$;
  - properties of limits in $\mathbb{R}/\overline{\mathbb{R}}$ (cluster point), compactness of closed and bounded intervals, compatibility of limit with arithmetic operations, the squeeze theorem.

This part is organized as follows. Chapter 7 contains some results from various fields of mathematics (set theory, algebraic structures, order theory, general topology, real and extended real numbers including second-countability), that are needed in the proofs of the integral theory. We recall that the material stated in [17] may also be used.

Then, Chapter 8 is devoted to subsets systems, from π-system to σ-algebra. Measurability and measurable space are presented in Chapter 9, and the specific case of sets of real and extended numbers is treated in Chapter 10. Chapter 11 is dedicated to measure and measure space, and Chapter 12 to the specific cases involving sets of numbers, including the construction of the Lebesgue measure on $\mathbb{R}$ through Carathéodory's extension theorem. Finally, the integral of nonnegative functions is addressed in Chapter 13 with for instance the Beppo Levi (monotone convergence) theorem, Fatou's lemma, and the Tonelli theorem. Chapter 14 is dedicated to the integral of functions with arbitrary sign, with the seminormed vector space $\mathcal{L}^1$ and Lebesgue's dominated convergence theorem.

Chapter 7

Complements

Contents

7.1 Complements on set theory

Definition 207 ***(pseudopartition).***
Let X be a set. Let $I \subset \mathbb{N}$. Subsets $(X_i)_{i\in I}$ of X are said to form a *pseudopartition (of X)* iff

$$\forall i, j \in I,\ i \neq j \Rightarrow X_i \cap X_j = \varnothing \quad \text{and} \quad X = \biguplus_{i\in I} X_i. \tag{7.1}$$

Remark 208. Note that, in contrast to proper partitions, parts of a pseudopartition may be empty.

Lemma 209 ***(compatibility of pseudopartition with intersection).*** *Let X be a set. Let $I \subset \mathbb{N}$. Let $A, (X_i)_{i\in I} \subset X$. Assume that $(X_i)_{i\in I}$ form a pseudopartition of X. Then, $(A \cap X_i)_{i\in I}$ form a pseudopartition of A, i.e. $A = \biguplus_{i\in I}(A \cap X_i)$.*

Proof. Direct consequence of Definition 207 (*pseudopartition*), and **distributivity of intersection over disjoint union**. □

Lemma 210 ***(technical inclusion for countable union).***
Let X be a set. Let $(A_n)_{n\in\mathbb{N}} \subset X$. Let $\varphi : \mathbb{N} \to \mathbb{N}$. Then, we have $\bigcup_{n\in\mathbb{N}} A_{\varphi(n)} \subset \bigcup_{n\in\mathbb{N}} A_n$.

Proof. Let $x \in \bigcup_{n\in\mathbb{N}} A_{\varphi(n)}$. then, from **the definition of union**, there exists $n \in \mathbb{N}$ such that $x \in A_{\varphi(n)}$. Thus, there exists $m = \varphi(n) \in \mathbb{N}$ such that $x \in A_m$. Hence, from **the definition of union**, we have $x \in \bigcup_{m\in\mathbb{N}} A_m$.

Therefore, we have $\bigcup_{n\in\mathbb{N}} A_{\varphi(n)} \subset \bigcup_{n\in\mathbb{N}} A_n$. □

Lemma 211 ***(order is meaningless in countable union).*** *Let X be a set. Let $(A_n)_{n\in\mathbb{N}} \subset X$. Let $\varphi : \mathbb{N} \to \mathbb{N}$. Assume that φ is bijective. Then, we have $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A_{\varphi(n)}$.*

Proof. Direct consequence of Lemma 210 (*technical inclusion for countable union*, with $(A_n)_{n\in\mathbb{N}}$ and φ, then $(A_{\varphi(n)})_{n\in\mathbb{N}}$ and φ^{-1} which satisfies $\varphi \circ \varphi^{-1} = \mathrm{Id}_{\mathbb{N}}$). □

Lemma 212 ***(definition of double countable union).***
Let X be a set. For all $n, m \in \mathbb{N}$, let $A_{n,m} \subset X$. Let $\varphi, \psi : \mathbb{N} \to \mathbb{N}^2$. Assume that φ and ψ are bijective. Then, $\bigcup_{p\in\mathbb{N}} A_{\varphi(p)} = \bigcup_{p\in\mathbb{N}} A_{\psi(p)}$. This union is denoted $\bigcup_{n,m\in\mathbb{N}} A_{n,m}$

Proof. Direct consequence of Lemma 211 (*order is meaningless in countable union*, with $(A_{\varphi(p)})_{p\in\mathbb{N}}$ and $\varphi^{-1} \circ \psi$). □

Lemma 213 ***(double countable union).*** *Let X be a set. For all $n, m \in \mathbb{N}$, let $A_{n,m} \subset X$. Then, we have $\bigcup_{n,m\in\mathbb{N}} A_{n,m} = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} A_{n,m}\right)$.*

Proof. From **countability of $\mathbb{N}^2$**, let $\varphi : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Then, from Lemma 212 (*definition of double countable union*), we have $\bigcup_{n,m\in\mathbb{N}} A_{n,m} = \bigcup_{p\in\mathbb{N}} A_{\varphi(p)}$ (and this does not depend on the choice for φ).

Let $x \in \bigcup_{p\in\mathbb{N}} A_{\varphi(p)}$. Then, from **the definition of union**, there exists $p_0 \in \mathbb{N}$ such that $x \in A_{\varphi(p_0)}$. Let $(n_0, m_0) \stackrel{\text{def.}}{=} \varphi(p_0)$. Thus, we have $x \in A_{n_0,m_0}$. Hence, from **the definition of union**, we have $x \in \bigcup_{m\in\mathbb{N}} A_{n_0,m} \subset \bigcup_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} A_{n,m}\right)$.

Conversely, let $x \in \bigcup_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} A_{n,m}\right)$. Then, from **the definition of union**, there exists $n_0 \in \mathbb{N}$ such that $x \in \bigcup_{m\in\mathbb{N}} A_{n_0,m}$, and there exists $m_0 \in \mathbb{N}$ such that $x \in A_{n_0,m_0}$. Let $p_0 \stackrel{\text{def.}}{=} \varphi^{-1}(n_0, m_0)$. Thus, we have $x \in A_{\varphi(p_0)}$. Hence, from **the definition of union**, we have $x \in \bigcup_{p\in\mathbb{N}} A_{\varphi(p)}$.

Therefore, we have $\bigcup_{n,m\in\mathbb{N}} A_{n,m} = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} A_{n,m}\right)$. □

Remark 214. The following lemma transforms a countable union of subsets into a countable disjoint union. It is achieved by adding the set differences of the subsets layer by layer. When the input sequence is nondecreasing, layers can be seen as onion peels.

Lemma 215 ***(partition of countable union).*** *Let X be a set. Let $(A_n)_{n\in\mathbb{N}} \subset X$. Let $B_0 \stackrel{\text{def.}}{=} A_0$, and for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i\in[0..n]} B_i$. Then, we have*

$$\forall m, n \in \mathbb{N}, \quad m \neq n \quad \Longrightarrow \quad B_m \cap B_n = \varnothing, \tag{7.2}$$

$$\forall n \in \mathbb{N}, \quad \bigcup_{i\in[0..n]} A_i = \biguplus_{i\in[0..n]} B_i, \tag{7.3}$$

$$\bigcup_{i\in\mathbb{N}} A_i = \biguplus_{i\in\mathbb{N}} B_i. \tag{7.4}$$

Proof. Let $p \in \mathbb{N}$. Then, from **the definition of set difference**, and **De Morgan's laws**,

$$B_{p+1} = A_{p+1} \cap \left(\bigcup_{i \in [0..p]} B_i \right)^c = A_{p+1} \cap \bigcap_{i \in [0..p]} B_i^c.$$

Let $i \in [0..p]$. Then, from **properties of set operations**, B_{p+1} and B_i are disjoint.

Let $m, n \in \mathbb{N}$. Assume that $m \neq n$, and let $i \stackrel{\text{def.}}{=} \min(m, n)$ and $p \stackrel{\text{def.}}{=} \max(m, n) - 1$. Then, we have $p \in \mathbb{N}$ and $i \in [0..p]$, and thus $B_m \cap B_n = B_{p+1} \cap B_i = \varnothing$.

For all $n \in \mathbb{N}$, let $P(n)$ be the property: $\bigcup_{i \in [0..n]} A_i = \biguplus_{i \in [0..n]} B_i$. **Induction: $P(0)$.** Trivial. **Induction: $P(n)$ implies $P(n+1)$.** Let $n \in \mathbb{N}$. Assume that $P(n)$ holds. Let $B \stackrel{\text{def.}}{=} \bigcup_{i \in [0..n]} B_i$. Then, from $P(n)$, and **properties of set operations (e.g., $A \cup B = A \uplus (B \setminus A)$)**, we have

$$\bigcup_{i \in [0..n+1]} A_i = B \cup A_{n+1} = B \uplus (A_{n+1} \setminus B) = B \uplus B_{n+1} = \biguplus_{i \in [0..n+1]} B_i.$$

Hence, for all $n \in \mathbb{N}$, $P(n)$ holds. Moreover, from **monotonicity of partial union**, and **the definition of the limit of monotone sequence of subsets**, we have $\bigcup_{i \in \mathbb{N}} A_i = \biguplus_{i \in \mathbb{N}} B_i$.

Therefore, all three properties hold. □

Definition 216 ***(trace of subsets of parties).*** Let X be a set. Let $Y \subset X$. Let i be the canonical injection from Y to X. Let $G \subset \mathcal{P}(X)$. The notation $G \overline{\cap} Y$ denotes the set

$$G \overline{\cap} Y \stackrel{\text{def.}}{=} i^{-1}(G) = \{A \cap Y \mid A \in G\}. \tag{7.5}$$

Definition 217 ***(product of subsets of parties).***
Let $m \in [2..\infty)$. For all $i \in [1..m]$, let X_i be a set, and $G_i \subset \mathcal{P}(X_i)$. Let $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$. The notations $G_1 \overline{\times} G_2$ and $\overline{\prod}_{i \in [1..m]} G_i$ respectively denote the sets

$$G_1 \overline{\times} G_2 \stackrel{\text{def.}}{=} \{A_1 \times A_2 \in \mathcal{P}(X_1 \times X_2) \mid A_1 \in G_1 \wedge A_2 \in G_2\} \tag{7.6}$$

$$\overline{\prod}_{i \in [1..m]} G_i \stackrel{\text{def.}}{=} \left\{ \prod_{i \in [1..m]} A_i \in \mathcal{P}(X) \;\middle|\; \forall i \in [1..m],\ A_i \in G_i \right\}. \tag{7.7}$$

Lemma 218 ***(restriction is masking).*** *Let X be a set. Let $A \subset Y \subset X$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, we have $\left(\hat{f} \mathbb{1}_A\right)_{|A} = f_{|A}$ and $\left(\hat{f} \mathbb{1}_A\right)_{|A^c} = 0$.*

Proof. Direct consequence of **the definition of the indicator function**, and **the definition of restriction of function**. □

7.2 Complements on algebraic structures

7.2.1 Vector space

Definition 219 *(relation compatible with vector operations).* Let $(E, +_E, \cdot_E)$ be a vector space. An equivalence relation $\mathcal{R}$ on E is said *compatible with the vector operations* iff

$$\forall u, u', v, v' \in E, \quad u \,\mathcal{R}\, u' \wedge v \,\mathcal{R}\, v' \Rightarrow (u +_E v) \,\mathcal{R}\, (u' +_E v'), \tag{7.8}$$

$$\forall \lambda \in \mathbb{K}, \forall u, u' \in E, \quad u \,\mathcal{R}\, u' \Rightarrow (\lambda \cdot_E u) \,\mathcal{R}\, (\lambda \cdot_E u'). \tag{7.9}$$

Lemma 220 *(quotient vector operations).* *Let $(E, +_E, \cdot_E)$ be a vector space. Let $\mathcal{R}$ be an equivalence relation on E. Assume that $\mathcal{R}$ is compatible with the vector operations. Then, the mappings $[+] : E/\mathcal{R} \times E/\mathcal{R} \to E/\mathcal{R}$ and $[\cdot] : \mathbb{K} \times E/\mathcal{R} \to E/\mathcal{R}$ defined by*

$$\forall u, v \in E, \quad [u] \,[+]\, [v] \stackrel{\text{def.}}{=} [u +_E v], \tag{7.10}$$

$$\forall \lambda \in \mathbb{K}, \forall u \in E, \quad \lambda \,[\cdot]\, [u] \stackrel{\text{def.}}{=} [\lambda \cdot_E u]. \tag{7.11}$$

are well-defined. These mappings are called quotient vector operations *induced on $E/\mathcal{R}$.*

Proof. Let $u, v \in E$. Let $\lambda \in \mathbb{K}$. Let $u', v' \in E$ such that $u \,\mathcal{R}\, u'$ and $v \,\mathcal{R}\, v'$, i.e. such that $[u'] = [u]$ and $[v'] = [v]$. Then, from Definition 219 (*relation compatible with vector operations*), we have $(u +_E v) \,\mathcal{R}\, (u' +_E v')$ and $(\lambda \cdot_E u) \,\mathcal{R}\, (\lambda \cdot_E u')$, i.e. $[u' +_E v'] = [u +_E v]$ and $[\lambda \cdot_E u'] = [\lambda \cdot_E u]$. Therefore, the quotient vector operations $[+]$ and $[\cdot]$ defined by Equations (7.10) and (7.11) do not depend on the choice of the representative of classes, i.e. they are well-defined. □

Lemma 221 *(quotient vector space, equivalence relation).*
Let $(E, +_E, \cdot_E)$ be a vector space. Let $\mathcal{R}$ be an equivalence relation on E. Assume that $\mathcal{R}$ is compatible with the vector operations. Let $[+]$ and $[\cdot]$ be the quotient vector operations induced on the quotient set $E/\mathcal{R}$. Then, $(E/\mathcal{R}, [+], [\cdot])$ is a vector space.

Proof. From **group theory**, $(E/\mathcal{R}, [+])$ is an abelian group with identity element $[0_E]$ Distributivity of the quotient scalar multiplication over quotient vector addition and field addition, compatibility of the quotient scalar multiplication with field multiplication, and 1 is the identity element for the quotient scalar multiplication are direct consequences of Definition 61 (*vector space*), and Lemma 220 (*quotient vector operations*). Therefore, from Definition 61 (*vector space*), $(E/\mathcal{R}, [+], [\cdot])$ is a vector space. □

Lemma 222 *(quotient vector space).* *Let E be a vector space. Let F be a vector subspace of E. Let $\mathcal{R}$ be the relation defined on E by for all $u, v \in E$, $u \,\mathcal{R}\, v \Leftrightarrow v - u \in F$.*
Then, $\mathcal{R}$ is an equivalence relation compatible with the vector operations of E.

The quotient set $E/\mathcal{R}$ equipped with the quotient vector operations is thus a vector space called quotient vector space of E by F*; it is denoted E/F. For all $u \in E$, the class of u is denoted $u + F$. The quotient vector operations are still denoted $+$ and $\cdot$ (the latter may be omitted).*

Proof. Let $u, u', v, v' \in E$. Let $\lambda \in \mathbb{K}$. Assume that $u \,\mathcal{R}\, u'$ and $v \,\mathcal{R}\, v'$, i.e. $u' - u, v' - v \in F$. Then, from Definition 61 (*vector space*, vector addition is associative, and scalar multiplication is distributive over field addition), and Lemma 81 (*closed under vector operations is subspace*), we have $(u' + v') - (u + v) = (u' - u) + (v' - v) \in F$ and $(\lambda u') - (\lambda u) = \lambda(u' - u) \in F$, i.e. $(u + v) \,\mathcal{R}\, (u' + v')$ and $(\lambda u) \,\mathcal{R}\, (\lambda u')$. Hence, from Definition 219 (*relation compatible with vector operations*), the equivalence relation $\mathcal{R}$ is compatible with the vector operations. Therefore, from Lemma 221 (*quotient vector space, equivalence relation*), the quotient set $E/\mathcal{R}$ equipped with the induced quotient vector operations is a vector space. □

Lemma 223 ***(linear map on quotient vector space).*** *Let E, F be vector spaces. Let G be a vector subspace of E. Let f be a linear map from E to F. Assume that $G \subset \ker(f)$. Then, the function $[f] \stackrel{\text{def.}}{=} (u + G \mapsto f(u))$ is a linear map from E/G to F.*

Proof. Let $u \in E$. Let $u' \in u + G$. Then, from hypotheses, we have $u' - u \in G \subset \ker(f)$. Thus, from Definition 64 (*linear map*), and Definition 101 (*kernel*), we have $f(u') = f(u)$. Hence, the function $[f]$ does not depend on the choice of the representative of equivalence classes, it is well-defined.

Let $(u + G), (v + G) \in E/G$. Let $\lambda \in \mathbb{K}$. Then, from Lemma 222 (*quotient vector space*), Lemma 220 (*quotient vector operations*), and Definition 64 (*linear map*), we have

$$[f](a(u + G)) = [f]((au) + G) = f(au) = af(u) = a[f](u + G),$$
$$[f]((u + G) + (v + G)) = [f]((u + v) + G) = f(u + v) = f(u) + f(v) = [f](u + G) + [f](v + G).$$

Therefore, from Definition 64 (*linear map*), $[f]$ is a linear map from E/G to F. □

7.2.2 Algebra over a field

Remark 224. The following algebraic structure "algebra over a field" is not to be confused with the concept of "set algebra" defined in Section 8.3.

Remark 225. Most results on algebras over a field are valid on algebras over a ring, for which the property of vector space over the field is replaced by that of module over the ring.

Definition 226 ***(algebra over a field).***
Let $\mathbb{K}$ be a field. A set E equipped with three algebra operations (a vector addition $+$, a scalar multiplication $\cdot$, and a vector multiplication $\times$), is called *algebra (over field $\mathbb{K}$)*, or $\mathbb{K}$*-algebra*, iff $(E, +, \cdot)$ is a $\mathbb{K}$-vector space, and vector multiplication is bilinear (or left and right distributive over vector addition, and compatible with scalars):

(7.12) $$\forall u, v, w \in E, \quad (u + v) \times w = (u \times w) + (v \times w),$$
(7.13) $$\forall u, v, w \in E, \quad u \times (v + w) = (u \times v) + (u \times w),$$
(7.14) $$\forall \lambda, \mu \in \mathbb{K}, \ \forall u, v \in E, \quad (\lambda \cdot u) \times (\mu \cdot v) = (\lambda\mu) \cdot (u \times v).$$

Remark 227. The $\cdot$ and $\times$ infix signs in the scalar and vector multiplications may be omitted.

Lemma 228 ***($\mathbb{K}$ is $\mathbb{K}$-algebra).*** *Let $\mathbb{K}$ be a field. It is an algebra over itself.*

Proof. Direct consequence of Definition 226 (*algebra over a field*), and **field properties of $\mathbb{K}$**. □

Definition 229 ***(inherited algebra operations).***
Let X be a nonempty set. Let $\mathbb{K}$ be a field. Let $(E, +_E, \cdot_E, \times_E)$ be an algebra over field $\mathbb{K}$. The *algebra operations inherited on E^X* are the mappings $+_{E^X}$ and $\cdot_{E^X}$ of Definition 91 (*inherited vector operations*), and the mapping $\times_{E^X} : E^X \times E^X \to E^X$ defined by

(7.15) $$\forall f, g \in E^X, \forall x \in X, \quad (f \times_{E^X} g)(x) \stackrel{\text{def.}}{=} f(x) \times_E g(x).$$

Remark 230. Usually, inherited algebra operations are denoted the same way as the algebra operations of the target algebra.

Lemma 231 ***(algebra of functions to algebra).***
Let $\mathbb{K}$ be a field. Let X be a nonempty set. Let $(E, +_E, \cdot_E, \times_E)$ be a $\mathbb{K}$-algebra. Let $+_{E^X}$, $\cdot_{E^X}$ and $\times_{E^X}$ be the algebra operations inherited on E^X. Then, $(E^X, +_{E^X}, \cdot_{E^X}, \times_{E^X})$ is a $\mathbb{K}$-algebra.

Proof. From Lemma 93 (*space of functions to space*), $(E^X, +_{E^X}, \cdot_{E^X})$ is a $\mathbb{K}$-vector space. Then, bilinearity of the inherited vector multiplication is a direct consequence of Definition 226 (*algebra over a field*), and Definition 229 (*inherited algebra operations*). Therefore, from Definition 226 (*algebra over a field*), E^X equipped with $+_{E^X}$, $\cdot_{E^X}$ and $\times_{E^X}$ is a $\mathbb{K}$-algebra. □

Lemma 232 *($\mathbb{K}^X$ is algebra).*
Let $\mathbb{K}$ be a field. Let X be a nonempty set. Then, $\mathbb{K}^X$ is a $\mathbb{K}$-algebra.

Proof. Direct consequence of Lemma 228 (*$\mathbb{K}$ is $\mathbb{K}$-algebra*), and Lemma 231 (*algebra of functions to algebra*). □

Definition 233 *(subalgebra).* Let $\mathbb{K}$ be a field. Let $(E, +, \cdot, \times)$ be a $\mathbb{K}$-algebra. A subset F of E equipped with the restrictions $+_{|F}$, $\cdot_{|F}$ and $\times_{|F}$ of the algebra operations to F is called *($\mathbb{K}$-)subalgebra of E* iff $(F, +_{|F}, \cdot_{|F}, \times_{|F})$ is a $\mathbb{K}$-algebra.

Remark 234. Usually, restrictions $+_{|F}$, $\cdot_{|F}$, and $\times_{|F}$ are still denoted $+$, $\cdot$ and $\times$.

Lemma 235 *(vector subspace and closed under multiplication is subalgebra).*
Let $\mathbb{K}$ be a field. Let $(E, +, \cdot, \times)$ be a $\mathbb{K}$-algebra. Let $F \subset E$. Then, F is a $\mathbb{K}$-subalgebra of E iff F is a $\mathbb{K}$-vector subspace of E, and F is closed under vector multiplication:

$$\forall u, v \in F, \quad u \times v \in F. \tag{7.16}$$

Proof. **"Left" implies "right".** Assume first that F is a $\mathbb{K}$-subalgebra of E. Then, from Definition 233 (*subalgebra*, $\underline{F \text{ is a } \mathbb{K}\text{-algebra}}$), Definition 226 (*algebra over a field*), $(F, +_{|F}, \cdot_{|F})$ is a $\mathbb{K}$-vector space, and F is closed under the restriction to F of the three operations. Thus, from Definition 77 (*subspace*), F is a $\mathbb{K}$-vector subspace of E, and F is closed under vector multiplication.

"Right" implies "left". Conversely, assume now that F is a $\mathbb{K}$-vector subspace of E, and that it is closed under vector multiplication. Then, from Definition 77 (*subspace*), $(F, +_{|F}, \cdot_{|F})$ is a $\mathbb{K}$-vector space. Moreover, from Lemma 81 (*closed under vector operations is subspace*), F is also closed under vector addition and scalar multiplication. Thus, since F is a subset of E, and E is a $\mathbb{K}$-algebra, Equations (7.12) to (7.14) are trivially satisfied over F with the restrictions of the three operations. Hence, from Definition 226 (*algebra over a field*), and Definition 233 (*subalgebra*), F is a $\mathbb{K}$-subalgebra of E.

Therefore, we have the equivalence. □

Lemma 236 *(closed under algebra operations is subalgebra).*
Let $\mathbb{K}$ be a field. Let E be a $\mathbb{K}$-algebra. Let $F \subset E$. Then, F is a $\mathbb{K}$-subalgebra of E iff $0_E \in F$, and F is closed under vector addition, scalar and vector multiplications:

$$\forall u, v \in F, \quad u + v \in F, \tag{7.17}$$

$$\forall \lambda \in \mathbb{K},\ \forall u \in F, \quad \lambda u \in F, \tag{7.18}$$

$$\forall u, v \in F, \quad u \times v \in F. \tag{7.19}$$

Proof. Direct consequence of Lemma 235 (*vector subspace and closed under multiplication is subalgebra*), and Lemma 81 (*closed under vector operations is subspace*). □

7.2.3 Seminormed vector space

Definition 237 ***(seminorm).***
Let $\mathbb{K}$ be a valued field. Let E be a $\mathbb{K}$-vector space. A function $\|\cdot\| : E \to \mathbb{R}$ is called *seminorm over* E iff it is absolutely homogeneous of degree 1, and it satisfies the triangle inequality:

$$\forall \lambda \in \mathbb{K},\ \forall u \in E, \quad \|\lambda u\| = |\lambda|\ \|u\|\,, \tag{7.20}$$

$$\forall u, v \in E, \quad \|u + v\| \leqslant \|u\| + \|v\|\,. \tag{7.21}$$

If so, $(E, \|\cdot\|)$ (or simply E) is called *seminormed ($\mathbb{K}$-)vector space.*

Remark 238. Most results from [17] on normed vector spaces can be generalized to the case of seminormed vector space, sometimes with slight modifications of the statement. In particular, the associated distance becomes a pseudometric.

Lemma 239 ***(definite seminorm is norm).*** *Let $(E, \|\cdot\|)$ be a seminormed vector space. Then, $(E, \|\cdot\|)$ is a normed vector space iff $\|\cdot\|$ is definite, i.e. for all $u \in E$, $\|u\| = 0 \Leftrightarrow u = 0$.*

Proof. Direct consequence of Definition 237 (*seminorm*), Definition 106 (*normed vector space*), and Definition 105 (*norm*). □

7.3 Complements on order theory

Remark 240. In the following definition, the strict inequality "$<$" naturally means "$\leqslant$ and not equal" (equality is related to the set object).

We recall the notations $\pm\infty$ to represent the extreme bounds of a totally ordered set; they may belong to the set, or not.

Definition 241 ***(interval).*** Let $(X, \leqslant)$ be a totally ordered nonempty set. Let $a, b \in X$.

The subset $\{x \in X \,|\, a < x < b\}$ is called the *open proper interval from a to b* (with excluded bounds a and b); it is denoted (a, b). The set of open proper intervals for all $a, b \in X$ is denoted $\mathcal{I}_X^{o,p}$.

The subsets $\{x \in X \,|\, x < b\}$ and $\{x \in X \,|\, a < x\}$ are called *open left ray and open right ray* (with excluded bounds a and b) (or *half-lines*); they are denoted $(a, \infty⟧$ and $⟦-\infty, b)$. The set of open rays for all $a, b \in X$ is denoted $\mathcal{R}_X^o$.

The notation $⟦-\infty, \infty⟧$ may be used to represent the whole set X.

In all cases, square brackets "$[,]$" are used to specify that bounds are included, and square-and-round brackets "$⟦,⟧$" are used to avoid specifying inclusion or exclusion of the bounds.

Let $a, b \in X \cup \{\pm\infty\}$. The subset $⟦a, b⟧$ is either a proper interval, a ray, or the whole set; it is called *interval*. The set of open intervals for all $a, b \in X \cup \{\pm\infty\}$ is denoted $\mathcal{I}_X^o \stackrel{\text{def.}}{=} \mathcal{I}_X^{o,p} \cup \mathcal{R}_X^o \cup \{X\}$.

Remark 242. Note that if X contains at least two elements, $\mathcal{I}_X^{o,p}$, $\mathcal{R}_X^o$ and $\{X\}$ are pairwise disjoint. In particular, (a, ∞) is an open ray when $\infty \notin X$, and an open proper interval when $\infty \in X$. More generally, open rays are never open proper intervals. For instance, when ∞ belongs to the set, $(a, \infty⟧$ is actually $(a, \infty]$ (an open subset of $\overline{\mathbb{R}}$), which is distinct from the open proper interval (a, ∞).

Lemma 243 ***(empty open interval).***
Let $(X, \leqslant)$ be a totally ordered nonempty set. Assume that X is dense-in-itself:

$$\forall x, y \in X, \quad x < y \quad \Longrightarrow \quad \exists z \in X,\ x < z < y. \tag{7.22}$$

Let $a, b \in X \cup \{\pm\infty\}$. Then, the open interval (a, b) is empty iff $b \leqslant a$.

Proof. Then, from Definition 241 (*interval*), Equation (7.22), **transitivity of order (which provides the other implication)**, and **the definition of strict inequality**, we have

$$\begin{aligned} (a, b) = \varnothing \quad &\Longleftrightarrow \quad \forall x \in X,\ \neg(a < x < b) \\ &\Longleftrightarrow \quad \neg(\exists x \in X,\ a < x < b) \quad \Longleftrightarrow \quad \neg(a < b) \quad \Longleftrightarrow \quad b \leqslant a. \end{aligned}$$

□

Remark 244. Of course, the previous statement is wrong in the presence of isolated points. For instance, $(0, 1)$ is empty in the discrete sets $\mathbb{N}$ and $\mathbb{Z}$.

Remark 245. In the following lemma, the left and right square-and-round brackets must remain the same on the sides of each interval. For instance "$⟦_1$" denotes either "$[$" or "$($", but it remains identical in the three intervals of Equation (7.23).

Lemma 246 ***(intervals are closed under finite intersection).***
Let $(X, \leqslant)$ be a totally ordered nonempty set. Then, the intersection of two proper intervals is a proper interval, the intersection of two rays is either a ray (if they both point towards the same direction), or a proper interval, and the intersection of a proper interval and a ray is a proper interval. In particular, for all $a, b, c, d \in X \cup \{\pm\infty\}$, we have (see remark above)

$$\llbracket_1 a, b\rrbracket_2 \cap \llbracket_1 c, d\rrbracket_2 = \llbracket_1 \max(a, c), \min(b, d)\rrbracket_2. \tag{7.23}$$

Hence, the closure of $\mathcal{R}_X^o \cup \{X\}$ under finite intersection is $\mathcal{I}_X^o$, and $\mathcal{I}_X^{o,p}$ and $\mathcal{I}_X^o$ are closed under finite intersection.

Proof. Direct consequence of Definition 241 (*interval*), **totally ordered set properties of X**, and **induction on the number of operands of the finite intersection**. □

Lemma 247 ***(empty intersection of open intervals).***
Let $(X, \leqslant)$ be a totally ordered nonempty set. Assume that X is dense-in-itself:

$$\forall x, y \in X, \quad x < y \quad \Longrightarrow \quad \exists z \in X,\ x < z < y. \tag{7.24}$$

Let $a, b, c, d \in X \cup \{\pm\infty\}$. Then, $(a, b) \cap (c, d)$ is empty iff $b \leqslant a$, $d \leqslant c$, $d \leqslant a$ or $b \leqslant c$.

Proof. Direct consequence of Lemma 246 (*intervals are closed under finite intersection*), Lemma 243 (*empty open interval*), and **totally ordered set properties of X**. □

7.4 Complements on general topology

Remark 248. In [17], we have only covered topology for metric spaces in which subsets are open when they contain a ball centered in each of their points. In the present document, we deal with the general case of topological spaces for which the collection of open subsets is given, e.g. via a topological basis.

Definition 249 ***(topological space).*** Let X be a set. A subset $\mathcal{T}$ of $\mathcal{P}(X)$ is called *topology of* X iff $\varnothing, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under (infinite) union and finite intersection.

If so, $(X, \mathcal{T})$ (or simply X) is called *topological space*, elements of $\mathcal{T}$ are called *open subsets of* X, and the complement of elements of $\mathcal{T}$ are called *closed subsets of* X.

Lemma 250 ***(intersection of topologies).***
Let X and I be sets. Let $(\mathcal{T}_i)_{i\in I}$ be topologies on X. Then, $\bigcap_{i\in I} \mathcal{T}_i$ is a topology on X.

Proof. Direct consequence of Definition 249 (*topological space*), and **the definition and properties of intersection and union of subsets**. □

Definition 251 ***(generated topology).*** Let X be a set. Let $G \subset \mathcal{P}(X)$. The *topology on X generated by G* is the intersection of all topologies on X containing G; it is denoted $\mathcal{T}_X(G)$. The generator G is also called *subbase of the topology*.

Lemma 252 ***(generated topology is minimum).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{T}_X(G)$ is the smallest topology on X containing G.

Proof. Direct consequence of Definition 251 (*generated topology*), Lemma 250 (*intersection of topologies*), and **properties of the intersection**. □

Lemma 253 ***(equivalent definition of generated topology).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Let $O \subset X$. Then, $O \in \mathcal{T}_X(G)$ iff O is the union of finite intersections of elements of $G \cup \{X\}$.

Proof. Direct consequence of Definition 249 (*topological space*), **associativity and commutativity of union**, **distributivity of intersection and union (both ways)** (thus, the set of unions of finite intersections of elements of G is a topology of X containing G, and contained in all topologies of X containing G), and Lemma 252 (*generated topology is minimum*). □

Definition 254 ***(topological basis).***
Let $(X, \mathcal{T})$ be a topological space. Let I be a set. A set $\{B_i \in \mathcal{T} \mid i \in I\}$ is called *topological basis of* $(X, \mathcal{T})$ iff for all $O \in \mathcal{T}$, there exists $J \subset I$ such that $O = \bigcup_{j\in J} B_j$.

Lemma 255 ***(augmented topological basis).*** *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Let $O \in \mathcal{T}$. Then, $\mathcal{B} \cup \{O\}$ is a topological basis of $(X, \mathcal{T})$.*

Proof. Direct consequence of Definition 254 (*topological basis*, with O open). □

Definition 256 ***(order topology).***
Let $(X, \leqslant)$ be a totally ordered set. The topology $\mathcal{T}_X(\mathcal{R}_X^o)$ is called *order topology on* X.

Remark 257. See Definition 241 (*interval*) for the definition of $\mathcal{R}_X^o$. Note that the order topology is the standard topology on the totally ordered sets of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Lemma 258 ***(topological basis of order topology).***
Let $(X, \leqslant)$ be a totally ordered set. Then, $\mathcal{I}_X^o$ is a topological basis for the order topology on X.

Proof. Direct consequence of Definition 256 (*order topology*), Lemma 253 (*equivalent definition of generated topology*), Lemma 246 (*intervals are closed under finite intersection*), and Definition 254 (*topological basis*). □

Remark 259. See Definition 241 (*interval*) for the definition of $\mathcal{I}_X^o$.

Lemma 260 ***(trace topology on subset).***
Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Let $Y \subset X$. Then, $\mathcal{T}_Y \stackrel{\text{def.}}{=} \mathcal{T} \overline{\cap} Y$ is a topology of Y, and $\mathcal{B} \overline{\cap} Y$ is a topological basis of $(Y, \mathcal{T}_Y)$.
$\mathcal{T}_Y$ is called trace topology, *and $(Y, \mathcal{T}_Y)$ is said* topological subspace of $(X, \mathcal{T})$.

Proof. Direct consequence of Definition 249 (*topological space*), Definition 254 (*topological basis*), Definition 216 (*trace of subsets of parties*), **distributivity of intersection over union**, and **commutativity of intersection**. □

Lemma 261 ***(box topology on Cartesian product).*** *Let I be a set. For all $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $\mathcal{B}_i$ be a topological basis of $(X_i, \mathcal{T}_i)$. Let $X \stackrel{\text{def.}}{=} \prod_{i\in I} X_i$. Then, $\mathcal{T} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in I} \mathcal{T}_i$ is a topology of X, and $\overline{\prod}_{i\in I} \mathcal{B}_i$ is a topological basis of $(X, \mathcal{T})$.*
$\mathcal{T}$ is called the box topology of X (induced by the $\mathcal{T}_i$'s).

Proof. Direct consequence of Definition 249 (*topological space*), Definition 254 (*topological basis*), Definition 217 (*product of subsets of parties*), and **distributivity of the Cartesian product over union**. □

7.4.1 Second axiom of countability

Definition 262 ***(second-countability).*** A topological space $(X, \mathcal{T})$ is said *second-countable*, or *completely separable*, iff it admits a countable topological basis,

Remark 263. This corresponds to the existence of $I \subset \mathbb{N}$ in Definition 254 (*topological basis*).

Lemma 264 ***(complete countable topological basis).***
Let $(X, \mathcal{T})$ be a second-countable topological space. Let $\mathcal{B}$ be a countable topological basis of $(X, \mathcal{T})$. Let $O \in \mathcal{T}$. Then, $\mathcal{B} \cup \{O\}$ is a countable topological basis of $(X, \mathcal{T})$.

Proof. Direct consequence of Definition 262 (*second-countability*), and Lemma 255 (*augmented topological basis*). □

Lemma 265 ***(compatibility of second-countability with Cartesian product).***
Let I be a set. For all $i \in I$, let $(X_i, \mathcal{T}_i)$ be a second-countable topological space, and let $\mathcal{B}_i$ be a countable topological basis of $(X_i, \mathcal{T}_i)$. Let $X \stackrel{\text{def.}}{=} \prod_{i\in I} X_i$. Assume that X is equipped with the box topology $\mathcal{T} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in I} \mathcal{T}_i$. Then, $\overline{\prod}_{i\in I} \mathcal{B}_i$ is a countable topological basis of $(X, \mathcal{T})$. Hence, $(X, \mathcal{T})$ is second-countable.

Proof. Direct consequence of Lemma 261 (*box topology on Cartesian product*), Definition 217 (*product of subsets of parties*), and **compatibility of finite Cartesian product with countability**. □

Lemma 266 ***(complete countable topological basis of product space).*** *Let I be a set. For all $i \in I$, let $(X_i, \mathcal{T}_i)$ be a second-countable topological space, let $\mathcal{B}_i$ be a countable topological basis of it, and let $O_i \in \mathcal{T}_i$. Let $X \stackrel{\text{def.}}{=} \prod_{i\in I} X_i$. Assume that X is equipped with the box topology $\mathcal{T} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in I} \mathcal{T}_i$. Then, $\overline{\prod}_{i\in I]} \mathcal{B}_i \cup \{O_i\}$ is a countable topological basis of $(X, \mathcal{T})$.*

Proof. Direct consequence of Definition 217 (*product of subsets of parties*), Lemma 264 (*complete countable topological basis*), and Lemma 261 (*box topology on Cartesian product*). □

7.4.2 Complements on metric space

Definition 267 ***(pseudometric).***
Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called *pseudometric over X* iff it is nonnegative, symmetric, it is zero on the diagonal, and it satisfies the triangle inequality:

$$\forall x, y \in X, \quad d(x, y) \geqslant 0, \tag{7.25}$$
$$\forall x, y \in X, \quad d(y, x) = d(x, y), \tag{7.26}$$
$$\forall x \in X, \quad d(x, x) = 0, \tag{7.27}$$
$$\forall x, y, z \in X, \quad d(x, z) \leqslant d(x, y) + d(y, z). \tag{7.28}$$

If so, (X, d) (or simply X) is called *pseudometric space.*

Remark 268. A pseudometric becomes a metric when it is also definite. Hence, most results from [17] on metric spaces are still valid on pseudometric spaces.

Lemma 269 ***(equivalent definition of convergent sequence).***
Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}, l \in X$. Then, $(x_n)_{n\in\mathbb{N}}$ is convergent with limit l iff

$$\forall k \in \mathbb{N},\ \exists N \in \mathbb{N},\ \forall n \in [N..\infty), \quad d(x_n, l) \leqslant \frac{1}{k+1}. \tag{7.29}$$

Proof. **"Left" implies "right".**
Direct consequence of Definition 27 (*convergent sequence*, with $\varepsilon \stackrel{\text{def.}}{=} \frac{1}{k+1}$).

"Right" implies "left". Assume that Equation (7.29) holds. Let $\varepsilon > 0$. Then, from **the Archimedean property of $\mathbb{R}$**, and **ordered field properties of $\mathbb{R}$**, let $k \in \mathbb{N}$ such that $k \geqslant \frac{1}{\varepsilon} - 1$, i.e. $\frac{1}{k+1} \leqslant \varepsilon$. Thus, from assumption, there exists $N \in \mathbb{N}$ such that for all $n \in [N..\infty)$, we have $d(x_n, l) \leqslant \frac{1}{k+1} \leqslant \varepsilon$. Hence, from Definition 27 (*convergent sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit l.

Therefore, we have the equivalence. □

Lemma 270 ***(convergent subsequence of Cauchy sequence).***
Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}} \in X$. Assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}$. Assume that $(n_k)_{k\in\mathbb{N}}$ is nondecreasing and that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is convergent. Then, $(x_n)_{n\in\mathbb{N}}$ is convergent with the same limit.

Proof. From Lemma 29 (*limit is unique*), let $x \in X$ be the limit of the subsequence. Let $\varepsilon > 0$. Then, from Definition 35 (*Cauchy sequence*), let $N \in \mathbb{N}$ such that for all $p, q \geqslant N$, we have $d(x_p, x_q) \leqslant \frac{\varepsilon}{2}$. Moreover, as $(n_k)_{k\in\mathbb{N}}$ is increasing, let $K' \in \mathbb{N}$ such that for all $k \geqslant K'$, we have $n_k > N$, and from Definition 27 (*convergent sequence*), let $K'' \in \mathbb{N}$ such that for all $k \geqslant K''$, we have $d(x_{n_k}, x) \leqslant \frac{\varepsilon}{2}$. Let $K \stackrel{\text{def.}}{=} \max(K', K'')$. Then, from Definition 17 (*distance*, triangle inequality), we have for all $n \geqslant N$,

$$d(x_n, x) \leqslant d(x_n, x_{n_K}) + d(x_{n_K}, x) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from Definition 27 (*convergent sequence*), $(x_n)_{n\in\mathbb{N}}$ is convergent with limit x. □

Definition 271 ***(cluster point).*** Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}} \in X$. A *cluster point* of the sequence is the limit x of any convergent subsequence of $(x_n)_{n\in\mathbb{N}}$:

$$\forall \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists n \geqslant N, \quad d(x_n, x) \leqslant \varepsilon. \tag{7.30}$$

7.5 Complements on real numbers

7.5.1 Real numbers

Lemma 272 ***(finite cover of compact interval).***
Let $a, b, (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{R}$. Assume that $a \leqslant b$ and $[a, b] \subset \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. Then, there exists $q \in \mathbb{N}$ and $(i_p)_{p\in[0..q]} \in \mathbb{N}$ pairwise distinct such that $[a, b] \subset \bigcup_{p\in[0..q]}(a_{i_p}, b_{i_p})$ with $a_{i_0} < a$, $b < b_{i_q}$, and for all $p \in [0..q-1]$, $a_{i_{p+1}} < b_{i_p}$.

Proof. From **the definition of compactness**, and **compactness of $[a, b]$**, there exists $n \in \mathbb{N}$ such that $[a, b] \subset \bigcup_{j\in[0..n]}(a_j, b_j)$.

Let $J \subset [0..n]$ and $x \in \mathbb{R}$. Assume that $[x, b] \subset \bigcup_{j\in J}(a_j, b_j)$ and $x \leqslant b$.

(1). Next index: $\exists i \overset{\text{def.}}{=} i_J(x) \in J,\ a_i < x < b_i$.
Direct consequence of **the definition of union (with $x \in [x, b]$)**.

(2). Next cover: $[b_i, b] \subset \bigcup_{j\in J\setminus\{i\}}(a_j, b_j)$.
Case $b < b_i$. Trivial. **Case $b_i \leqslant b$.** Then, from (1), we have $b_i \in (x, b]$. Let $y \in [b_i, b] \subset [x, b]$. Then, from **the definition of union**, there exists $j \in J$ such that $a_j < y < b_j$. Hence, from **transitivity of order in $\mathbb{R}$**, we have $b_i < b_j$, i.e. $j \neq i$, and $[b_i, b] \subset \bigcup_{j\in J\setminus\{i\}}(a_j, b_j)$.

Let $i_{-1} \overset{\text{def.}}{=} -1$, $J_{-1} \overset{\text{def.}}{=} [0..n]$, and $b_{i_{-1}} = b_{-1} \overset{\text{def.}}{=} a$. Let $(i_p)_{p\in[0..q]}$ be the sequence of integers computed by the following algorithm:

$p \overset{\text{def.}}{=} -1$;
repeat
  $i_{p+1} \overset{\text{def.}}{=} i_{J_p}(b_{i_p})$;
  $J_{p+1} \overset{\text{def.}}{=} J_p \setminus \{i_{p+1}\}$;
  $p \overset{\text{def.}}{=} p + 1$;
until $b < b_{i_p}$;
return $(i_0, \ldots, i_p)$;

For all $p \in \{-1\} \cup \mathbb{N}$, let $I_p \overset{\text{def.}}{=} \{i_0, \ldots, i_p\}$ (with $I_{-1} \overset{\text{def.}}{=} \varnothing$), and $P(p)$ be the property:

$$\operatorname{card}(I_p) = p + 1 \quad \wedge \quad [0..n] = I_p \uplus J_p \quad \wedge \quad (\forall m \in [0..p],\ a_{i_m} < b_{i_{m-1}}) \quad \wedge$$
$$[a, b_{i_p}) \subset \bigcup_{j\in I_p}(a_j, b_j) \quad \wedge \quad [b_{i_p}, b] \subset \bigcup_{j\in J_p}(a_j, b_j).$$

Let us show that there exists $q \in [0..n]$ such that $P(q) \wedge b < b_{i_q}$ holds.

(3). Initialization: $P(-1) \wedge b_{i_{-1}} \leqslant b$. Trivial.

(4). Iterations: $\forall p \in \{-1\} \cup \mathbb{N},\ P(p) \wedge b_{i_p} \leqslant b$ implies $P(p + 1)$.
Let $p \in \{-1\} \cup \mathbb{N}$. Assume that $P(p) \wedge b_{i_p} \leqslant b$ holds.

Let $J \overset{\text{def.}}{=} J_p$ and $x \overset{\text{def.}}{=} b_{i_p}$. Then, we have $[x, b] \subset \bigcup_{j\in J}(a_j, b_i)$ and $x \leqslant b$. Thus, from (1), there exists $i_{p+1} = i_J(x) \in J = J_p$ (i.e. $i_{p+1} \notin I_p$), such that $a_{i_{p+1}} < b_{i_p} < b_{i_{p+1}}$. Moreover, from (2), we have $[b_{i_{p+1}}, b] \subset \bigcup_{j \in J_{p+1}} (a_j, b_j)$. Then, we have

$$\begin{gathered}
\operatorname{card}(I_{p+1}) = \operatorname{card}(I_p) + 1 = p + 2, \\
[0..n] = I_p \uplus \{i_{p+1}\} \uplus (J_p \setminus \{i_{p+1}\}) = I_{p+1} \uplus J_{p+1}. \\
a_{i_{p+1}} < b_{i_p}, \\
[a, b_{i_{p+1}}) = [a, b_{i_p}) \uplus [b_{i_p}, b_{i_{p+1}}) \subset \bigcup_{j \in I_p} (a_j, b_j) \cup (a_{i_{p+1}}, b_{i_{p+1}}) = \bigcup_{j \in I_{p+1}} (a_j, b_j), \\
[b_{i_{p+1}}, b] \subset \bigcup_{j \in J_{p+1}} (a_j, b_j).
\end{gathered}$$

Hence, $P(p+1)$ holds.

(5). Termination: $\exists q \in [0..n],\ P(q) \wedge b < b_{i_q}$. Assume that $P(p) \wedge b_{i_p} \leqslant b$ holds for all p in $[0..n]$. Then, from (4), we have $P(n+1)$. Thus, from **additivity of the cardinality**, we have

$$n + 1 = \operatorname{card}([0..n]) = \operatorname{card}(I_{n+1}) + \operatorname{card}(J_{p+1}) \geqslant \operatorname{card}(I_{n+1}) = n + 2.$$

Which is impossible. Hence, there exists $q \in [0..n]$ such that $P(q) \wedge b < b_{i_q}$ holds.
Moreover, we have $[a, b] \subset [a, b_{i_q}) \subset \bigcup_{j \in I_q} (a_j, b_j)$.

Therefore, there exists $q \in \mathbb{N}$ and $(i_p)_{p \in [0..q]} \in \mathbb{N}$ pairwise distinct such that $[a, b]$ is included in $\bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$ with $a_{i_0} < a$, $b < b_{i_q}$, and for all $p \in [0..q-1]$, $a_{i_{p+1}} < b_{i_p}$. □

Definition 273 ***(Hölder conjugates in $\mathbb{R}$).***
Real numbers $p, q \in (1, \infty)$ are said *Hölder conjugates in* $\mathbb{R}$ iff $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 274 ***(2 is self-Hölder conjugate in $\mathbb{R}$).***
The real number 2 is Hölder conjugate in $\mathbb{R}$ with itself.

Proof. Direct consequence of Definition 273 (*Hölder conjugates in* $\mathbb{R}$), and since $\frac{1}{2} + \frac{1}{2} = 1$. □

Lemma 275 ***(Young's inequality for products in $\mathbb{R}$).*** *Let $p, q \in (1, \infty)$. Assume that p and q are Hölder conjugates. Let $a, b \in \mathbb{R}_+$. Then, we have $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof. **Case $ab = 0$.** Then, from **nonnegativeness of exponentiation**, **closedness of multiplicative inverse in $\mathbb{R}_+^\star$, and of the multiplication and addition in $\mathbb{R}_+$**, The right-hand side is nonnegative. hence, the inequality holds.

Case $ab > 0$. Then, from **the zero-product property in $\mathbb{R}_+$ (contrapositive)**, we have $a, b > 0$. Thus, from **algebraic properties of the natural logarithm function**, Definition 273 (*Hölder conjugates in* $\mathbb{R}$), and **concavity of the natural logarithm function (with $\frac{1}{p} + \frac{1}{q} = 1$)**, we have

$$\ln(ab) = \ln a + \ln b = \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) \leqslant \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right).$$

Hence, from **monotonicity of the exponential function**, and since **the exponential and logarithm functions in $\mathbb{R}$ are each other inverse**, we have $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.

Therefore, the inequality always holds. □

Lemma 276 ***(Young's inequality for products in $\mathbb{R}$, case $p = 2$).***
Let $a, b \in \mathbb{R}_+$. Let $\varepsilon > 0$. Then, we have $ab \leqslant \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$.

Proof. Direct consequence of Lemma 274 (*2 is self-Hölder conjugate in* $\mathbb{R}$), Lemma 275 (*Young's inequality for products in* $\mathbb{R}$, with $p = q \overset{\text{def.}}{=} 2$ and $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon} b \in \mathbb{R}_+$), and **properties of square root and multiplicative inverse in $\mathbb{R}_+^\star$, and of multiplication in $\mathbb{R}_+$.** □

Remark 277. Note that a similar result also holds in the general case of Hölder conjugate numbers $p, q \in (1, \infty)$: for all $a, b \in \mathbb{R}_+$, for all $\varepsilon > 0$, we have $ab \leqslant \frac{a^p}{p \varepsilon^{\frac{2}{q}}} + \frac{\varepsilon^{\frac{2}{p}} b^q}{q}$.

7.5.2 Extended real numbers

Definition 278 ***(extended real numbers, $\overline{\mathbb{R}}$).*** The set of *extended real numbers* is $\overline{\mathbb{R}} \overset{\text{def.}}{=} \mathbb{R} \uplus \{-\infty, \infty\}$, and the order in $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ with the following rule:

$$\text{1.} \quad \forall a \in \mathbb{R}, \quad -\infty < a < \infty. \tag{7.31}$$

Lemma 279 ***(order in $\overline{\mathbb{R}}$ is total).*** *$(\overline{\mathbb{R}}, \leqslant)$ is a totally ordered set.*

Proof. Direct consequence of **the definition of total order**, Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*), and **totality of order in** $\mathbb{R}$. ☐

Remark 280. The goal of this section is to clarify some properties of basic operations such as addition, multiplication, and exponentiation, when extended to $\overline{\mathbb{R}}$.

Remark 281. Note that, as with regular real numbers, all results on operations on extended real numbers can be lifted into similar results on functions taking their values in $\overline{\mathbb{R}}$.

Definition 282 ***(addition in $\overline{\mathbb{R}}$).***
Addition and subtraction in $\mathbb{R}$ are extended to $\overline{\mathbb{R}}$ with the following rules:

$$\begin{cases} \text{1.} \quad \forall a > -\infty, \quad a + \infty = \infty + a \overset{\text{def.}}{=} \infty, \\ \text{2.} \quad \forall a < \infty, \quad a + (-\infty) = -\infty + a \overset{\text{def.}}{=} -\infty, \\ \text{3.} \quad \infty + (-\infty) \text{ and } -\infty + \infty \text{ are undefined,} \\ \text{4.} \quad -(\pm\infty) = \mp\infty, \\ \text{5.} \quad \forall a, b \in \overline{\mathbb{R}}, \quad a + (-b) \text{ defined} \quad \Longrightarrow \quad a - b \overset{\text{def.}}{=} a + (-b). \end{cases} \tag{7.32}$$

Lemma 283 ***(zero is identity element for addition in $\overline{\mathbb{R}}$).***
Let $a \in \overline{\mathbb{R}}$. Then, we have $a + 0 = 0 + a = a$.

Proof. Direct consequence of **abelian group properties of $(\mathbb{R}, +)$**, and Definition 282 (*addition in $\overline{\mathbb{R}}$*, new rules 1, and 2 are compatible with the property). ☐

Lemma 284 ***(addition in $\overline{\mathbb{R}}$ is associative when defined).***
Let $a, b, c \in \overline{\mathbb{R}}$. Then, $a + (b + c)$ and $(a + b) + c$ are either equal or both undefined.

Proof. Direct consequence of **associativity of addition in** $\mathbb{R}$, and Definition 282 (*addition in $\overline{\mathbb{R}}$*, new rules 1, 2, and 3 are compatible with associativity). ☐

Lemma 285 ***(addition in $\overline{\mathbb{R}}$ is commutative when defined).***
Let $a, b \in \overline{\mathbb{R}}$. Then, $a + b$ and $b + a$ are either equal or both undefined.

Proof. Direct consequence of **commutativity of addition in** $\mathbb{R}$, and Definition 282 (*addition in $\overline{\mathbb{R}}$*, new rules 1, 2, and 3 are compatible with commutativity). ☐

Lemma 286 ***(infinity-sum property in $\overline{\mathbb{R}}$).*** *Let $a, b \in \overline{\mathbb{R}}$. Then, we have*

$$a + b = \infty \iff (a = \infty \quad \land \quad b > -\infty) \quad \lor \quad (a > -\infty \quad \land \quad b = \infty) \tag{7.33}$$

$$a + b = -\infty \iff (a = -\infty \quad \land \quad b < \infty) \quad \lor \quad (a < \infty \quad \land \quad b = -\infty). \tag{7.34}$$

Moreover, if we assume that the sum $a + b$ is well-defined, then we have

$$a + b = \pm\infty \iff a = \pm\infty \quad \lor \quad b = \pm\infty. \tag{7.35}$$

Proof. Direct consequence of Definition 282 (*addition in* $\overline{\mathbb{R}}$, rules 1, 2 and 3). □

Lemma 287 *(additive inverse in* $\overline{\mathbb{R}}$ *is monotone).*
Let $a, b \in \overline{\mathbb{R}}$*. Then, we have* $a \leqslant b$ *iff* $-b \leqslant -a$*.*

Proof. Direct consequence of Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$), **monotonicity of additive inverse in** $\mathbb{R}$, and Definition 282 (*addition in* $\overline{\mathbb{R}}$, rule 4). □

Definition 288 *(multiplication in* $\overline{\mathbb{R}}$*).*
Multiplication and division by nonzero in $\mathbb{R}$ are extended to $\overline{\mathbb{R}}$ with the following rules:

$$\begin{cases} 1. \quad \forall a > 0, \quad a \times (\pm\infty) = \pm\infty \times a \overset{\text{def.}}{=} \pm\infty, \\ 2. \quad \forall a < 0, \quad a \times (\pm\infty) = \pm\infty \times a \overset{\text{def.}}{=} \mp\infty, \\ 3. \quad 0 \times (\pm\infty) \text{ and } \pm\infty \times 0 \text{ are undefined}, \\ 4. \quad \frac{1}{0} \text{ is undefined, and } \frac{1}{\pm\infty} \overset{\text{def.}}{=} 0, \\ 5. \quad \forall a, b \in \overline{\mathbb{R}}, \quad \frac{1}{b} \text{ and } a \times \frac{1}{b} \text{ defined} \quad \Longrightarrow \quad \frac{a}{b} \overset{\text{def.}}{=} a \times \frac{1}{b}. \end{cases} \tag{7.36}$$

Remark 289. The 5th rule in the previous definition implies that $\frac{\infty}{\pm\infty}$, $\frac{-\infty}{\pm\infty}$, and $\frac{a}{0}$ (for all $a \in \overline{\mathbb{R}}$) are undefined. Note that rule 3 is modified in the context of measure theory in Definition 333.

Lemma 290 *(multiplication in* $\overline{\mathbb{R}}$ *is associative when defined).*
Let $a, b, c \in \overline{\mathbb{R}}$*. Then,* $a \times (b \times c)$ *and* $(a \times b) \times c$ *are either equal or both undefined.*

Proof. Direct consequence of **associativity of multiplication in** $\mathbb{R}$, and Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, new rules 1, 2, and 3 are compatible with associativity). □

Lemma 291 *(multiplication in* $\overline{\mathbb{R}}$ *is commutative when defined).*
Let $a, b \in \overline{\mathbb{R}}$*. Then,* $a \times b$ *and* $b \times a$ *are either equal or both undefined.*

Proof. Direct consequence of **commutativity of multiplication in** $\mathbb{R}$, and Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, new rules 1, 2, and 3 are compatible with commutativity). □

Lemma 292 *(multiplication in* $\overline{\mathbb{R}}$ *is left distributive over addition when defined).*
Let $a, b, c \in \overline{\mathbb{R}}$*. Then,* $a \times (b + c)$ *and* $(a \times b) + (a \times c)$ *are either equal or both undefined.*

Proof. Direct consequence of **left distributivity of multiplication over addition in** $\mathbb{R}$, Definition 282 (*addition in* $\overline{\mathbb{R}}$), and Definition 288 (*multiplication in* $\overline{\mathbb{R}}$) (new rules are compatible with left distributivity of multiplication over addition). □

Lemma 293 *(multiplication in* $\overline{\mathbb{R}}$ *is right distributive over addition when defined).*
Let $a, b, c \in \overline{\mathbb{R}}$*. Then,* $(a + b) \times c$ *and* $(a \times c) + (b \times c)$ *are either equal or both undefined.*

Proof. Direct consequence of Lemma 291 (*multiplication in* $\overline{\mathbb{R}}$ *is commutative when defined*, used twice), and Lemma 292 (*multiplication in* $\overline{\mathbb{R}}$ *is left distributive over addition when defined*). □

Lemma 294 *(zero-product property in* $\overline{\mathbb{R}}$*).* *Let* $a, b \in \overline{\mathbb{R}}$*. Then, we have*

$$ab = 0 \iff ab \text{ is defined} \quad \wedge \quad (a = 0 \quad \vee \quad b = 0). \tag{7.37}$$

Proof. Direct consequence of **the zero-product property in** $\mathbb{R}$, and Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rule 3). □

Lemma 295 ***(infinity-product property in*** $\overline{\mathbb{R}}$***).*** *Let* $a, b \in \overline{\mathbb{R}}$*. Then, we have*

$$ab = \infty \iff (a = \infty \;\wedge\; b > 0) \;\vee\; (a = -\infty \;\wedge\; b < 0) \;\vee\; (a > 0 \;\wedge\; b = \infty) \;\vee\; (a < 0 \;\wedge\; b = -\infty), \tag{7.38}$$

$$ab = -\infty \iff (a = -\infty \;\wedge\; b > 0) \;\vee\; (a = \infty \;\wedge\; b < 0) \;\vee\; (a > 0 \;\wedge\; b = -\infty) \;\vee\; (a < 0 \;\wedge\; b = \infty). \tag{7.39}$$

Proof. **"Left" implies "right".** Assume that $ab = \pm\infty$. Then, from Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rules 1 and 2), we have $a = \pm\infty$ or $b = \pm\infty$. Assume that the other operand is zero. Then, from Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rule 3), the product ab is undefined. Which is impossible.

Assume that $ab = \infty$. Then, from Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rules 1 and 2), we have either $a = \infty$ and $b > 0$, $a = -\infty$ and $b < 0$, $b = \infty$ and $a > 0$, or $b = -\infty$ and $a < 0$.

Similarly, assume now that $ab = -\infty$. Then, again from Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rules 1 and 2), we have either $a = -\infty$ and $b > 0$, $a = \infty$ and $b < 0$, $b = -\infty$ and $a > 0$, or $b = \infty$ and $a < 0$.

"Right" implies "left". Direct consequence of Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rules 1 and 2).

Therefore, we have the equivalences. □

Lemma 296 ***(finite-product property in*** $\overline{\mathbb{R}}$***).*** *Let* $a, b \in \overline{\mathbb{R}}$*. Then, we have*

$$ab \textit{ is defined} \;\wedge\; ab \in \mathbb{R} \iff a, b \in \mathbb{R}. \tag{7.40}$$

Proof. Direct consequence of Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, new rules are not finite-product rules). □

Definition 297 ***(absolute value in*** $\overline{\mathbb{R}}$***).***
The absolute value in $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ with the following rule:

$$|\pm\infty| = \infty. \tag{7.41}$$

Lemma 298 ***(equivalent definition of absolute value in*** $\overline{\mathbb{R}}$***).***
Let $a \in \overline{\mathbb{R}}$*. Then, we have* $|a| = \max(-a, a)$*.*

Proof. Direct consequence of **the definition of absolute value in** $\mathbb{R}$, and Definition 297 (*absolute value in* $\overline{\mathbb{R}}$). □

Lemma 299 ***(bounded absolute value in*** $\overline{\mathbb{R}}$***).***
Let $a, b \in \overline{\mathbb{R}}$*. Then, we have* $|a| \leqslant b$ *iff* $-b \leqslant a \leqslant b$*.*

Proof. Direct consequence of Lemma 298 (*equivalent definition of absolute value in* $\overline{\mathbb{R}}$, $-a \leqslant b$ and $a \leqslant b$), and Lemma 287 (*additive inverse in* $\overline{\mathbb{R}}$ *is monotone*, $-b \leqslant a$). □

Lemma 300 ***(bounded absolute value in*** $\overline{\mathbb{R}}$ ***(strict)).***
Let $a, b \in \overline{\mathbb{R}}$*. Then, we have* $|a| < b$ *iff* $-b < a < b$*.*

Proof. Direct consequence of Lemma 298 (*equivalent definition of absolute value in* $\overline{\mathbb{R}}$, $-a < b$ and $a < b$), and Lemma 287 (*additive inverse in* $\overline{\mathbb{R}}$ *is monotone*, contrapositive both ways, $-b < a$). □

Lemma 301 ***(finite absolute value in*** $\overline{\mathbb{R}}$***).*** *Let* $a \in \overline{\mathbb{R}}$*. Then, we have* $|a|$ *finite iff* a *finite.*

Proof. Direct consequence of Lemma 300 (*bounded absolute value in* $\overline{\mathbb{R}}$ *(strict)*, with $b \overset{\text{def.}}{=} \infty$). □

Lemma 302 ***(absolute value in $\overline{\mathbb{R}}$ is nonnegative).*** *The absolute value in $\overline{\mathbb{R}}$ is nonnegative.*

Proof. Direct consequence of Definition 297 (*absolute value in $\overline{\mathbb{R}}$*), **nonnegativeness of the absolute value in $\mathbb{R}$.** and **nonnegativeness of ∞**. □

Lemma 303 ***(absolute value in $\overline{\mathbb{R}}$ is even).*** *The absolute value in $\overline{\mathbb{R}}$ is even.*

Proof. Direct consequence of **evenness of the absolute value in $\mathbb{R}$.** and Definition 297 (*absolute value in $\overline{\mathbb{R}}$*, new rule is compatible with evenness), □

Lemma 304 ***(absolute value in $\overline{\mathbb{R}}$ is definite).*** *The absolute value in $\overline{\mathbb{R}}$ is definite.*

Proof. Direct consequence of **definiteness of the absolute value in $\mathbb{R}$**, and Definition 297 (*absolute value in $\overline{\mathbb{R}}$*, new rule is not a zero-absolute-value rule). □

Lemma 305 ***(absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality).***
Let $a, b \in \overline{\mathbb{R}}$. Assume that $a + b$ is well-defined. Then, we have $|a + b| \leqslant |a| + |b|$.

Proof. Direct consequence of Definition 297 (*absolute value in $\overline{\mathbb{R}}$*), **the triangle inequality for the absolute value in $\mathbb{R}$**, Definition 282 (*addition in $\overline{\mathbb{R}}$*, ∞ is absorbing for addition in $\overline{\mathbb{R}}_+$), and Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*, ∞ is the maximal element). □

Definition 306 ***(exponential and logarithm in $\overline{\mathbb{R}}$).***
The exponential function in $\mathbb{R}$ and the natural logarithm function in $\mathbb{R}_+^\star$ are respectively extended to $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ with the following rules:

$$\exp(-\infty) = 0, \quad \exp\infty = \infty, \quad \ln 0 = -\infty, \quad \text{and} \quad \ln\infty = \infty. \tag{7.42}$$

Lemma 307 ***(exponential and logarithm in $\overline{\mathbb{R}}$ are inverse).***
The exponential function in $\overline{\mathbb{R}}$ and the natural logarithm function in $\overline{\mathbb{R}}_+$ are each other inverse.

Proof. Direct consequence of **properties of the exponential and natural logarithm functions in $\mathbb{R}$**, and Definition 306 (*exponential and logarithm in $\overline{\mathbb{R}}$*, new rules are compatible with the property). □

Definition 308 ***(exponentiation in $\overline{\mathbb{R}}$).*** Exponentiation in $\mathbb{R}$ (with either positive base and any exponent, or zero base and positive exponent) is extended to $\overline{\mathbb{R}}$ with the following rule:

$$\forall a \in \overline{\mathbb{R}}_+,\ \forall b \in \overline{\mathbb{R}}, \quad b\ln a \text{ defined} \quad \Longrightarrow \quad a^b \overset{\text{def.}}{=} \exp(b\ln a). \tag{7.43}$$

Lemma 309 ***(exponentiation in $\overline{\mathbb{R}}$).***
Exponentiation in $\overline{\mathbb{R}}$ is the function $\exp : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}_+$ defined by

$$\begin{cases} 1. \quad \forall a \in \mathbb{R}_+^\star,\ \forall b \in \mathbb{R}, \quad a^b \overset{\text{def.}}{=} \exp(b\ln a) \in \mathbb{R}_+^\star, \\ 2. \quad \forall a \in [0,1),\ \forall b > 0, \quad a^\infty = 0^b \overset{\text{def.}}{=} 0 \quad \wedge \quad a^{-\infty} = \infty^b \overset{\text{def.}}{=} \infty, \\ 3. \quad \forall a \in (1,\infty],\ \forall b < 0, \quad a^\infty = 0^b \overset{\text{def.}}{=} \infty \quad \wedge \quad a^{-\infty} = \infty^b \overset{\text{def.}}{=} 0, \\ 4. \quad 0^0, \infty^0, \text{ and } 1^{\pm\infty} \text{ are undefined.} \end{cases} \tag{7.44}$$

Proof. Direct consequence of **the definition of exponentiation in $\mathbb{R}$**, Definition 308 (*exponentiation in $\overline{\mathbb{R}}$*), Definition 306 (*exponential and logarithm in $\overline{\mathbb{R}}$*), and Definition 288 (*multiplication in $\overline{\mathbb{R}}$*). □

Lemma 310 ***(topology of*** $\overline{\mathbb{R}}$***).***
The totally ordered set $(\overline{\mathbb{R}}, \leqslant)$ *is equipped with the order topology* $\mathcal{T}_{\overline{\mathbb{R}}}(\mathcal{R}^o_{\overline{\mathbb{R}}})$ *generated by the open rays, for which the open intervals* $\mathcal{I}^o_{\overline{\mathbb{R}}}$ *constitute a topological basis.*

Proof. Direct consequence of Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$), Definition 256 (*order topology*), and Lemma 258 (*topological basis of order topology*). □

Remark 311. Note that $\overline{\mathbb{R}}$ is actually metrizable since it is homeomorphic to a bounded segment, e.g. $[-\pi/2, \pi/2]$ or $[-1, 1]$. Indeed, metrics can be defined on $\overline{\mathbb{R}}$, for instance by using the arctangent function, or the hyperbolic tangent function.

Lemma 312 ***(trace topology on*** $\mathbb{R}$***).***
Let $\overline{\mathcal{T}}$ *be the order topology on* $\overline{\mathbb{R}}$*. Then, the trace topology* $\overline{\mathcal{T}}_{\mathbb{R}}$ *is equal to the order topology on* $\mathbb{R}$*.*

Proof. Direct consequence of Lemma 260 (*trace topology on subset*), Lemma 258 (*topological basis of order topology*, since $\mathbb{R} \in \mathcal{I}^o_{\overline{\mathbb{R}}} \subset \mathcal{I}^o_{\overline{\mathbb{R}}}$), and Definition 254 (*topological basis*). □

Remark 313. As a consequence, open subsets of $\mathbb{R}$ are also open subsets of $\overline{\mathbb{R}}$.

Lemma 314 ***(convergence towards*** $-\infty$***).***
Let $(x_n)_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$*. Then, we have* $\lim_{n\to\infty} x_n = -\infty$ *iff*

$$\forall k \in \mathbb{N},\ \exists N \in \mathbb{N},\ \forall n \in [N..\infty), \quad x_n \leqslant -k. \tag{7.45}$$

If so, the sequence is said convergent towards $-\infty$*.*

Proof. Direct consequence of Lemma 310 (*topology of* $\overline{\mathbb{R}}$), **the definition of the limit using neighborhoods**, **the Archimedean property of** $\mathbb{R}$, and **totally ordered set properties of** $\mathbb{R}$. □

Lemma 315 ***(continuity of addition in*** $\overline{\mathbb{R}}$***).*** *Addition in* $\overline{\mathbb{R}}$ *is continuous when defined.*

Proof. Direct consequence of **continuity of addition in** $\mathbb{R}$. **unboundedness of addition when an operand tends towards** $\pm\infty$, and Definition 282 (*addition in* $\overline{\mathbb{R}}$, rules 1 and 2). □

Lemma 316 ***(continuity of multiplication in*** $\overline{\mathbb{R}}$***).***
Multiplication in $\overline{\mathbb{R}}$ *is continuous when defined.*

Proof. Direct consequence of **continuity of multiplication in** $\mathbb{R}$. **unboundedness of multiplication when an operand tends towards** $\pm\infty$ **and the other is nonzero**, and Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rules 1 and 2). □

Lemma 317 ***(absolute value in*** $\overline{\mathbb{R}}$ ***is continuous).*** *The absolute value in* $\overline{\mathbb{R}}$ *is continuous.*

Proof. Direct consequence of **continuity of the absolute value in** $\mathbb{R}$, and Definition 297 (*absolute value in* $\overline{\mathbb{R}}$, absolute value is closed in $\{\pm\infty\}$). □

7.5.2.1 Nonnegative extended real numbers

Lemma 318 ***(addition in*** $\overline{\mathbb{R}}_+$ ***is closed).*** *Addition in* $\overline{\mathbb{R}}_+$ *is closed.*

Proof. Direct consequence of **closedness of addition in** $\mathbb{R}_+$, and Definition 282 (*addition in* $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur). □

Lemma 319 ***(addition in $\overline{\mathbb{R}}_+$ is associative).*** *Addition in $\overline{\mathbb{R}}_+$ is associative.*

Proof. Direct consequence of Lemma 284 (*addition in $\overline{\mathbb{R}}$ is associative when defined*), and Definition 282 (*addition in $\overline{\mathbb{R}}$*, undefined forms of rule 3 cannot occur). □

Lemma 320 ***(addition in $\overline{\mathbb{R}}_+$ is commutative).*** *Addition in $\overline{\mathbb{R}}_+$ is commutative.*

Proof. Direct consequence of Lemma 285 (*addition in $\overline{\mathbb{R}}$ is commutative when defined*), and Definition 282 (*addition in $\overline{\mathbb{R}}$*, undefined forms of rule 3 cannot occur). □

Lemma 321 ***(infinity-sum property in $\overline{\mathbb{R}}_+$).*** *Let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$a + b = \infty \quad \Longleftrightarrow \quad a = \infty \quad \vee \quad b = \infty. \tag{7.46}$$

Proof. Direct consequence of Lemma 286 (*infinity-sum property in $\overline{\mathbb{R}}$*). □

Lemma 322 ***(series are convergent in $\overline{\mathbb{R}}_+$).***
Let $(a_n)_{n\in\mathbb{N}} \in \overline{\mathbb{R}}_+$. Then, we have $\sum_{n\in\mathbb{N}} a_n \in \overline{\mathbb{R}}_+$.

Proof. Direct consequence of **completeness of $\overline{\mathbb{R}}_+$**. □

Lemma 323 ***(technical upper bound in series in $\overline{\mathbb{R}}_+$).*** *Let $(a_p)_{p\in\mathbb{N}} \in \overline{\mathbb{R}}_+$. Let $\varphi : \mathbb{N} \to \mathbb{N}$. Assume that φ is injective. Then, we have $\sum_{j\in\mathbb{N}} a_{\varphi(j)} \leqslant \sum_{p\in\mathbb{N}} a_p$.*

Proof. Let

$$A = \sum_{j\in\mathbb{N}} a_{\varphi(j)} \stackrel{\text{def.}}{=} \lim_{i\to\infty} \sum_{j\in[0..i]} a_{\varphi(j)} \quad \text{and} \quad B = \sum_{p\in\mathbb{N}} a_p \stackrel{\text{def.}}{=} \lim_{n\to\infty} \sum_{p\in[0..n]} a_p.$$

Then, from Lemma 322 (*series are convergent in $\overline{\mathbb{R}}_+$*), we have $A, B \in \overline{\mathbb{R}}_+$. Let $i \in \mathbb{N}$. Let

$$n \stackrel{\text{def.}}{=} \max\{\varphi(j) \,|\, j \in [0..i]\}.$$

Then, from **the definition of the maximum**, we have $\varphi([0..i]) \subset [0..n] \subset \mathbb{N}$. Thus, from **totally ordered set properties of $\overline{\mathbb{R}}_+$**, we have

$$\sum_{j\in[0..i]} a_{\varphi(j)} \leqslant \sum_{p\in[0..n]} a_p \leqslant B.$$

Hence, from **monotonicity of the limit (when $n \to \infty$)**, we have $A \leqslant B$.
Therefore, we have $\sum_{j\in\mathbb{N}} a_{\varphi(j)} \leqslant \sum_{p\in\mathbb{N}} a_p$. □

Lemma 324 ***(order is meaningless in series in $\overline{\mathbb{R}}_+$).***
Let $(a_p)_{p\in\mathbb{N}} \in \overline{\mathbb{R}}_+$. Let $\varphi : \mathbb{N} \to \mathbb{N}$. Assume that φ is bijective. Then, $\sum_{p\in\mathbb{N}} a_p = \sum_{j\in\mathbb{N}} a_{\varphi(j)}$.

Proof. Direct consequence of Lemma 323 (*technical upper bound in series in $\overline{\mathbb{R}}_+$*, first used with the sequence $(a_p)_{n\in\mathbb{N}}$ and the function φ, and then with the sequence $(b_j)_{j\in\mathbb{N}} \stackrel{\text{def.}}{=} (a_{\varphi(j)})_{j\in\mathbb{N}}$ and the function φ^{-1} which satisfies $\varphi \circ \varphi^{-1} = \mathrm{Id}_{\mathbb{N}}$). □

Lemma 325 ***(definition of double series in $\overline{\mathbb{R}}_+$).***
Let $(a_{p,q})_{p,q\in\mathbb{N}} \in \overline{\mathbb{R}}_+$. Let $\varphi, \psi : \mathbb{N} \to \mathbb{N}^2$. Assume that φ and ψ are bijections.
Then, we have $\sum_{j\in\mathbb{N}} a_{\varphi(j)} = \sum_{j\in\mathbb{N}} a_{\psi(j)}$. This sum is denoted $\sum_{p,q\in\mathbb{N}} a_{p,q}$.

Proof. Direct consequence of Lemma 324 (*order is meaningless in series in* $\overline{\mathbb{R}}_+$, with $(a_{\varphi(j)})_{j\in\mathbb{N}}$ and $\varphi^{-1}\circ\psi$). □

Lemma 326 ***(double series in*** $\overline{\mathbb{R}}_+$***).***
Let $(a_{p,q})_{p,q\in\mathbb{N}} \in \overline{\mathbb{R}}_+$. *Then, we have* $\sum_{p,q\in\mathbb{N}} a_{p,q} = \sum_{p\in\mathbb{N}}\left(\sum_{q\in\mathbb{N}} a_{p,q}\right)$.

Proof. From **countability of** $\mathbb{N}^2$, let $\varphi : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Then, from Lemma 325 (*definition of double series in* $\overline{\mathbb{R}}_+$), let $A \stackrel{\text{def.}}{=} \sum_{p,q\in\mathbb{N}} = \sum_{j\in\mathbb{N}} a_{\varphi(j)}$ (the sum does not depend on the choice for φ). Let $B \stackrel{\text{def.}}{=} \sum_{p\in\mathbb{N}}\left(\sum_{q\in\mathbb{N}} a_{p,q}\right)$. Then, from Lemma 322 (*series are convergent in* $\overline{\mathbb{R}}_+$), we have $A, B \in \overline{\mathbb{R}}_+$.

Let $i \in \mathbb{N}$. Let $n \stackrel{\text{def.}}{=} \max\{\pi_1(\varphi(j)) \,|\, j \in [0..i]\}$ and $m \stackrel{\text{def.}}{=} \max\{\pi_2(\varphi(j)) \,|\, j \in [0..i]\}$ where the functions π_1 and π_2 are defined by $\forall p, q \in \mathbb{N}$, $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$. Then, from **the definition of the maximum**, we have $\varphi([0..i]) \subset [0..n] \times [0..m] \subset \mathbb{N}^2$. Thus, from **totally ordered set properties of** $\overline{\mathbb{R}}_+$, we have

$$\sum_{j\in[0..i]} a_{\varphi(j)} \leqslant \sum_{p\in[0..n]}\left(\sum_{p\in[0..m]} a_{p,q}\right) \leqslant B.$$

Hence, from **monotonicity of the limit (when** $i \to \infty$**)**, we have $A \leqslant B$.

Let $n, m \in \mathbb{N}$. Let $i \stackrel{\text{def.}}{=} \max\{\varphi^{-1}(p, q) \,|\, (p, q) \in [0..n] \times [0..m]\}$. Then, from **the definition of the maximum**, we have $[0..n] \times [0..m] \subset \varphi([0..i]) = \mathbb{N}^2$. Thus, from **totally ordered set properties of** $\overline{\mathbb{R}}_+$, we have

$$\sum_{p\in[0..n]}\left(\sum_{q\in[0..m]} a_{p,q}\right) \leqslant \sum_{j\in[0..i]} a_{\varphi(j)} \leqslant A.$$

Hence, from **monotonicity of the limit (when** $m \to \infty$**, then** $n \to \infty$**)**, we have $B \leqslant A$.

Therefore, we have $\sum_{p,q\in\mathbb{N}} = \sum_{p\in\mathbb{N}}\left(\sum_{q\in\mathbb{N}} a_{p,q}\right)$. □

Definition 327 ***(multiplication in*** $\overline{\mathbb{R}}_+$***).***
Multiplication and division in $\overline{\mathbb{R}}$ are restricted to $\overline{\mathbb{R}}_+$ by replacing the fourth rule in (7.36) by

$$4'.\quad \frac{1}{0} \stackrel{\text{def.}}{=} \infty, \text{ and } \frac{1}{\infty} \stackrel{\text{def.}}{=} 0. \tag{7.47}$$

Remark 328. When restricting to nonnegative extended numbers, making multiplicative inverse a bijection from $\overline{\mathbb{R}}_+$ onto itself through Definition 327 implies that for all $a > 0$, $\frac{a}{0} \stackrel{\text{def.}}{=} \infty$. The expressions $0 \times \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ remain undefined.

Lemma 329 ***(multiplication in*** $\overline{\mathbb{R}}_+$ ***is closed when defined).***
When they are defined, multiplication and division in $\overline{\mathbb{R}}_+$ *are closed.*

Proof. Direct consequence of **closedness of multiplication and division by nonzero in** $\mathbb{R}_+$, Definition 288 (*multiplication in* $\overline{\mathbb{R}}$), and Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$). □

Lemma 330 ***(zero-product property in*** $\overline{\mathbb{R}}_+$***).*** *Let* $a, b \in \overline{\mathbb{R}}_+$. *Then, we have*

$$ab = 0 \iff ab \text{ is defined} \quad\wedge\quad (a = 0 \quad\vee\quad b = 0). \tag{7.48}$$

Proof. Direct consequence of Lemma 294 (*zero-product property in* $\overline{\mathbb{R}}$), and Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$, new rule is not a zero-product rule). □

Lemma 331 ***(infinity-product property in $\overline{\mathbb{R}}_+$).*** *Let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$ab = \infty \iff (a = \infty \quad \wedge \quad b \neq 0) \quad \vee \quad (a \neq 0 \quad \wedge \quad b = \infty). \tag{7.49}$$

Proof. Direct consequence of Lemma 295 (*infinity-product property in $\overline{\mathbb{R}}$*), and Definition 327 (*multiplication in $\overline{\mathbb{R}}_+$*, new rule is not an infinity-product rule). $\square$

Lemma 332 ***(finite-product property in $\overline{\mathbb{R}}_+$).*** *Let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$ab \text{ is defined} \quad \wedge \quad ab \in \mathbb{R}_+ \iff a, b \in \mathbb{R}_+. \tag{7.50}$$

Proof. Direct consequence of Lemma 296 (*finite-product property in $\overline{\mathbb{R}}$*), **closedness of multiplication in $\mathbb{R}_+$**, and Definition 327 (*multiplication in $\overline{\mathbb{R}}_+$*, new rule is not a finite-product rule). $\square$

7.5.2.2 In the context of measure theory

Definition 333 ***(multiplication in $\overline{\mathbb{R}}$ (measure theory)).***
In the context of measure theory (and probability), the third rule in (7.36) is replaced by

$$3'. \quad 0 \times (\pm\infty) = \pm\infty \times 0 \stackrel{\text{def.}}{=} 0. \tag{7.51}$$

Remark 334. In the context of measure theory, making 0 an absorbing element for multiplication in $\overline{\mathbb{R}}$ through the previous definition implies that $\frac{\infty}{\pm\infty} = \frac{-\infty}{\pm\infty} \stackrel{\text{def.}}{=} 0$. The expression $\frac{a}{0}$ (for all $a \in \overline{\mathbb{R}}$) remains undefined. Note that multiplication in $\overline{\mathbb{R}}$ is always well-defined in this context.

Lemma 335 ***(zero-product property in $\overline{\mathbb{R}}$ (measure theory)).***
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$ab = 0 \iff a = 0 \quad \vee \quad b = 0. \tag{7.52}$$

Proof. Direct consequence of Lemma 294 (*zero-product property in $\overline{\mathbb{R}}$*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, new rule $3'$ is compatible with the property). $\square$

Lemma 336 ***(infinity-product property in $\overline{\mathbb{R}}$ (measure theory)).***
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$\begin{aligned} ab = \infty \iff & (a = \infty \quad \wedge \quad b > 0) \quad \vee \quad (a = -\infty \quad \wedge \quad b < 0) \quad \vee \\ & (a > 0 \quad \wedge \quad b = \infty) \quad \vee \quad (a < 0 \quad \wedge \quad b = -\infty), \end{aligned} \tag{7.53}$$

$$\begin{aligned} ab = -\infty \iff & (a = -\infty \quad \wedge \quad b > 0) \quad \vee \quad (a = \infty \quad \wedge \quad b < 0) \quad \vee \\ & (a > 0 \quad \wedge \quad b = -\infty) \quad \vee \quad (a < 0 \quad \wedge \quad b = \infty). \end{aligned} \tag{7.54}$$

Proof. Direct consequence of Lemma 295 (*infinity-product property in $\overline{\mathbb{R}}$*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, new rule is not an infinity-product rule). $\square$

Lemma 337 ***(finite-product property in $\overline{\mathbb{R}}$ (measure theory)).***
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$ab \in \mathbb{R} \iff a, b \in \mathbb{R} \quad \vee \quad a = 0 \quad \vee \quad b = 0. \tag{7.55}$$

Proof. Direct consequence of Lemma 336 (*infinity-product property in $\overline{\mathbb{R}}$ (measure theory)*, contrapositive), **De Morgan's laws**, and **distributivity of logical conjunction over logical disjunction**. $\square$

Lemma 338 ***(multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)).*** *In the context of measure theory, multiplication and division are total functions $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$. In particular, we have $\frac{\infty}{\infty} = \frac{0}{0} \stackrel{\text{def.}}{=} 0$, and for all $a > 0$, $\frac{a}{0} \stackrel{\text{def.}}{=} \infty$.*

Proof. Direct consequence of Definition 288 (*multiplication in $\overline{\mathbb{R}}$*), Definition 327 (*multiplication in $\overline{\mathbb{R}}_+$*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*). ☐

Remark 339. Of course, in the context of measure theory, multiplication and division are no longer continuous on the whole boundary of $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$.

Lemma 340 ***(multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)).*** *In the context of measure theory, multiplication in $\overline{\mathbb{R}}_+$ is associative.*

Proof. Direct consequence of Lemma 290 (*multiplication in $\overline{\mathbb{R}}$ is associative when defined*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, no longer undefined forms). ☐

Lemma 341 ***(multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)).*** *In the context of measure theory, multiplication in $\overline{\mathbb{R}}_+$ is commutative.*

Proof. Direct consequence of Lemma 291 (*multiplication in $\overline{\mathbb{R}}$ is commutative when defined*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, no longer undefined forms). ☐

Lemma 342 ***(multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)).*** *In the context of measure theory, multiplication is (left and right) distributive over addition in $\overline{\mathbb{R}}_+$.*

Proof. Direct consequence of Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*), Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, no longer undefined forms). ☐

Lemma 343 ***(zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)).*** *In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$ab = 0 \iff a = 0 \quad \vee \quad b = 0. \tag{7.56}$$

Proof. Direct consequence of Lemma 330 (*zero-product property in $\overline{\mathbb{R}}_+$*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, new rule 3′ is compatible with the property). ☐

Lemma 344 ***(infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)).*** *In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$ab = \infty \iff (a = \infty \quad \wedge \quad b > 0) \quad \vee \quad (b = \infty \quad \wedge \quad a > 0). \tag{7.57}$$

Proof. Direct consequence of Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*), and Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*, new rule is not an infinity-product rule). ☐

Lemma 345 ***(finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)).*** *In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_+$. Then, we have*

$$ab \in \mathbb{R}_+ \iff a, b \in \mathbb{R}_+ \quad \vee \quad a = 0 \quad \vee \quad b = 0. \tag{7.58}$$

Proof. Direct consequence of Lemma 337 (*finite-product property in $\overline{\mathbb{R}}$ (measure theory)*), and **closedness of multiplication in $\mathbb{R}_+$**. ☐

Lemma 346 ***(exponentiation in*** $\overline{\mathbb{R}}$ ***(measure theory)).***
In the context of measure theory, the fourth rule in (7.44) is replaced by

$$4'.\quad 0^0 = \infty^0 = 1^{\pm\infty} \stackrel{\text{def.}}{=} 1. \tag{7.59}$$

Thus, we have

$$\forall a \in \overline{\mathbb{R}}_+,\ a^0 = 1 \quad \text{and} \quad \forall b \in \overline{\mathbb{R}},\ 1^b = 1. \tag{7.60}$$

Proof. Direct consequence of Definition 308 (*exponentiation in* $\overline{\mathbb{R}}$), Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*), and Lemma 309 (*exponentiation in* $\overline{\mathbb{R}}$, new rule only affects rule 4). □

Definition 347 ***(Hölder conjugates).***
Extended numbers $p, q \in [1, \infty]$ are said *Hölder conjugates* iff $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 348. The previous definition implies that extended numbers 1 and ∞ are Hölder conjugate, since from rule $4'$ of Definition 327, we have $\frac{1}{\infty} = 0$.

Lemma 349 ***(Young's inequality for products (measure theory)).***
In the context of measure theory, let $p, q \in (1, \infty)$. Assume that p and q are Hölder conjugates. Let $a, b \in \overline{\mathbb{R}}_+$. Then, we have $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$.

Proof. From Lemma 338 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed (measure theory)*), Lemma 309 (*exponentiation in* $\overline{\mathbb{R}}$, with $p, q \neq 0, \pm\infty$), **closedness of multiplicative inverse in** $\mathbb{R}_+^\star$, Lemma 329 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed when defined*), Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rule 5 with $b = p, q \in \mathbb{R}_+^\star$), and Lemma 318 (*addition in* $\overline{\mathbb{R}}_+$ *is closed*), ab, a^p, b^q, $\frac{a^p}{p}$, $\frac{b^q}{q}$, and $\frac{a^p}{p} + \frac{b^q}{q}$ are well-defined, and belong to $\overline{\mathbb{R}}_+$.

Case $ab = \infty$. Then, from Lemma 344 (*infinity-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*, $a = \infty$ or $b = \infty$), Lemma 309 (*exponentiation in* $\overline{\mathbb{R}}$, rule 2 with $b = p, q > 0$), Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, rule 1 with $a \stackrel{\text{def.}}{=} \frac{1}{p}, \frac{1}{q} > 0$), and Definition 282 (*addition in* $\overline{\mathbb{R}}$, rule 1, the other operand is not $-\infty$), the right-hand side equals ∞. Thus, the (in)equality holds.

Case $ab = 0$. Then, the inequality holds since the right-hand side is nonnegative.

Case $ab \in \mathbb{R}_+^\star$. Direct consequence of Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*, contrapositive), Lemma 331 (*infinity-product property in* $\overline{\mathbb{R}}_+$, contrapositive), and Lemma 275 (*Young's inequality for products in* $\mathbb{R}$).

Therefore, the inequality always holds. □

Lemma 350 ***(Young's inequality for products, case $p = 2$ (measure theory)).***
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_+$. Let $\varepsilon \in \mathbb{R}_+^\star$. Then, we have $ab \leqslant \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$.

Proof. Direct consequence of Lemma 274 (*2 is self-Hölder conjugate in* $\mathbb{R}$), Lemma 349 (*Young's inequality for products (measure theory)*, with $p = q \stackrel{\text{def.}}{=} 2$ and $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon} b \in \overline{\mathbb{R}}_+$), **properties of square root and multiplicative inverse in** $\mathbb{R}_+^\star$, Lemma 338 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed (measure theory)*), Lemma 340 (*multiplication in* $\overline{\mathbb{R}}_+$ *is associative (measure theory)*), and Lemma 341 (*multiplication in* $\overline{\mathbb{R}}_+$ *is commutative (measure theory)*). □

7.5.3 Second countability and real numbers

Definition 351 ***(connected component in $\mathbb{R}$).***
Let $A \subset \mathbb{R}$. Let $x \in A$. The *connected component of A containing x* is the union of all open intervals I containing x and included in A; it is denoted I_x^A.

Lemma 352 ***(connected component of open subset of $\mathbb{R}$ is open interval).***
Let $O \subset \mathbb{R}$ be open. Let $x \in O$. Then, I_x^O is an open interval contained in O.

Proof. Note that from Definition 22 (*open subset*), there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset O$. Then, from Definition 351 (*connected component in $\mathbb{R}$*), $(x-\varepsilon, x+\varepsilon) \subset I_x^O$. Thus, I_x^O is nonempty, contains x. Moreover, from Definition 249 (*topological space*, closedness under union), I_x^O is open.

Let $a, b, c \in \mathbb{R}$ such that $a < c < b$ and $a, b \in I_x^O$. Then, from Definition 351 (*connected component in $\mathbb{R}$*), and **the definition of union**, there exists intervals $I_a(x)$ and $I_b(x)$ such that $x, a \in I_a(x) \subset I_x^O \subset O$ and $x, b \in I_b(x) \subset I_x^O \subset O$. **Case $c = x$.** Then, we have $c = x \in I_x^O$. **Case $c < x$.** Then, we have $a < c < x$. Thus, since $a, x \in I_a(x)$, and $I_a(x)$ is an interval, we have successively $c \in (a, x) \subset I_a(x) \subset I_x^O$. **Case $x < c$.** Then, we have $x < c < b$. Thus, since $x, b \in I_b(x)$, and $I_b(x)$ is an interval, we have successively $c \in (x, b) \subset I_b(x) \subset I_x^O$. Hence, we always have $c \in I_x^O$. Thus, from Definition 241 (*interval*, with $X \overset{\text{def.}}{=} \mathbb{R}$), I_x^O is an interval.

Therefore, I_x^O is an open interval contained in O. □

Lemma 353 ***(connected component of open subset of $\mathbb{R}$ is maximal).***
Let $O \subset \mathbb{R}$ be open. Let $x, y \in O$. Assume that $y \in I_x^O$. Then, we have $I_y^O = I_x^O$.

Proof. From Definition 351 (*connected component in $\mathbb{R}$*, union), there exists an open interval I such that $y \in I$ and $x \in I \subset O$. Thus, we also have $x \in I_y^O$. But, from Lemma 352 (*connected component of open subset of $\mathbb{R}$ is open interval*), I_x^O is an open interval such that $y \in I_x^O \subset O$, and I_y^O is an open interval such that $x \in I_y^O \subset O$. Thus, from Definition 351 (*connected component in $\mathbb{R}$*), we have $I_x^O \subset I_y^O$ and $I_y^O \subset I_x^O$. Therefore, we have $I_x^O = I_y^O$. □

Lemma 354 ***(connected components of open subset of $\mathbb{R}$ are equal or disjoint).***
Let $O \subset \mathbb{R}$ be open. Let $x, y \in O$. Then, we have either $I_x^O = I_y^O$, or $I_x^O \cap I_y^O = \varnothing$.

Proof. Assume that $I_x^O \cap I_y^O \neq \varnothing$. Then, there exists $z \in I_x^O \cap I_y^O$. Thus, from Lemma 353 (*connected component of open subset of $\mathbb{R}$ is maximal*), we have $I_z^O = I_x^O$ and $I_z^O = I_y^O$. Therefore, I_x^O and I_y^O are equal or disjoint. □

Theorem 355 ***(countable connected components of open subsets of $\mathbb{R}$).***
Let $O \subset \mathbb{R}$ be open. Then, O is a countable union of disjoint open intervals.

Proof. Let $x \in O$. Then, from Definition 22 (*open subset*), there exists $\varepsilon > 0$ such that the open interval $(x-\varepsilon, x+\varepsilon)$ is included in O. From **density of rational numbers in $\mathbb{R}$**, there exists $q \in \mathbb{Q} \cap (x-\varepsilon, x+\varepsilon)$. Then, by construction, we have $x \in (x-\varepsilon, x+\varepsilon) \subset I_q^O$. Thus, we have the inclusion $O \subset \bigcup_{q \in \mathbb{Q} \cap O} I_q^O$.

Conversely, let $q \in \mathbb{Q} \cap O$. Then, from Definition 351 (*connected component in $\mathbb{R}$*), we have $I_q^O \subset O$. Thus, we also have the other inclusion $\bigcup_{q \in \mathbb{Q} \cap O} I_q^O \subset O$. Hence, the equality.

Therefore, from **countability of $\mathbb{Q}$**, Lemma 354 (*connected components of open subset of $\mathbb{R}$ are equal or disjoint*), and after eliminating doubles in the union, O is a countable union of disjoint open intervals. □

Lemma 356 ***(rational approximation of lower bound of open interval).***
Let $a, b \in \overline{\mathbb{R}}$. Assume that $a < b$. Then, there exists a sequence $(a_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$ that is nonincreasing with limit a.

Proof. Note that $-\infty \leqslant a < b \leqslant \infty$. From **density of rational numbers in** $\mathbb{R}$, let $a_0 \in (a,b)\cap\mathbb{Q}$.

Case $a = -\infty$. Then, from **density of rational numbers in** $\mathbb{R}$, for all $n \in \mathbb{N}$, let a_{n+1} be in $(-\infty, -2|a_n|) \cap \mathbb{Q}$. Hence, from **ordered field properties of** $\mathbb{R}$, and **the Archimedean property of** $\mathbb{R}$, the sequence $(a_n)_{n\in\mathbb{N}}$ belongs to (a,b), and is nonincreasing with limit $a = -\infty$.

Case a finite. Then, from **density of rational numbers in** $\mathbb{R}$, for all $n \in \mathbb{N}$, let a_{n+1} be in $\left(a, \frac{a+a_n}{2}\right) \cap \mathbb{Q} \subset (a,b) \cap \mathbb{Q}$. Let $n \in \mathbb{N}$. Then, from **ordered field properties of** $\mathbb{R}$, we have $a_{n+1} < a_n$ and $0 < a_{n+1} - a < \frac{a_n - a}{2}$. Thus, a trivial finite induction on index i shows that

$$\forall i \in [0..n+1], \quad 0 < a_{n+1} - a < \frac{a_i - a}{2^{n+1-i}} < \frac{a_0 - a}{2^{n+1}}.$$

Hence, from **the squeeze theorem**, the sequence $(a_n)_{n\in\mathbb{N}}$ is nonincreasing with limit a. □

Lemma 357 ***(rational approximation of upper bound of open interval).*** *Let $a, b \in \overline{\mathbb{R}}$. Assume that $a < b$. Then, there exists a sequence $(b_n)_{n\in\mathbb{N}} \in \mathbb{Q} \cap (a,b)$ that is nonincreasing with limit b.*

Proof. Direct consequence of Lemma 356 (*rational approximation of lower bound of open interval*, with $a \stackrel{\text{def.}}{=} -b$ and $b \stackrel{\text{def.}}{=} -a$). □

Lemma 358 ***(open intervals with rational bounds cover open interval).*** *Let $a, b \in \overline{\mathbb{R}}$. Assume that $a < b$. Then, there exists $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that $(a,b) = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$.*

Proof. Direct consequence of Lemma 356 (*rational approximation of lower bound of open interval*, there exists $(a_n)_{n\in\mathbb{N}} \in \mathbb{Q} \cap (a,b)$), Lemma 357 (*rational approximation of upper bound of open interval*, there exists $(b_n)_{n\in\mathbb{N}} \in \mathbb{Q} \cap (a,b)$), Definition 27 (*convergent sequence*, with $\varepsilon \stackrel{\text{def.}}{=} x - a$, then $\varepsilon \stackrel{\text{def.}}{=} b - x$ for all $x \in (a,b)$ (hence, $(a,b) \subset \bigcup_{n\in\mathbb{N}}(a_n,b_n)$)), and Definition 249 (*topological space*, closedness under union ($\bigcup_{n\in\mathbb{N}}(a_n,b_n) \subset (a,b)$)). □

Theorem 359 ***($\mathbb{R}$ is second-countable).*** *Let d be the Euclidean distance on $\mathbb{R}$. Then, $\{(a,b)\,|\,a,b \in \mathbb{Q} \wedge a < b\}$ is a topological basis of $(\mathbb{R}, d)$. Hence, $(\mathbb{R}, d)$ is second-countable.*

Proof. Direct consequence of Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$), Lemma 358 (*open intervals with rational bounds cover open interval*), Definition 254 (*topological basis*), **countability of** $\mathbb{Q} \times \mathbb{Q}_+^\star$, and Definition 262 (*second-countability*). □

Lemma 360 ***($\mathbb{R}^n$ is second-countable).*** *Let $n \in [2..\infty)$. Let d be the Euclidean distance on $\mathbb{R}^n$. Then, $\left\{ \prod_{i\in[1..n]}(a_i,b_i) \,\middle|\, \forall i \in [1..n], a_i, b_i \in \mathbb{Q} \wedge a_i < b_i \right\}$ is a topological basis of $(\mathbb{R}^n, d)$. Hence, $(\mathbb{R}^n, d)$ is second-countable.*

Proof. Direct consequence of Theorem 359 ($\mathbb{R}$ *is second-countable*), Lemma 265 (*compatibility of second-countability with Cartesian product*), Lemma 261 (*box topology on Cartesian product*) and Definition 262 (*second-countability*). □

Lemma 361 ***(open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$).*** *Let $a, b \in \overline{\mathbb{R}}$. Assume that $a < b$. Then, there exists $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that*

$$(a,b) = \bigcup_{n\in\mathbb{N}}(a_n,b_n), \quad [-\infty, b) = \bigcup_{n\in\mathbb{N}}[-\infty, b_n) \quad \text{and} \quad (a,\infty] = \bigcup_{n\in\mathbb{N}}(a_n,\infty]. \tag{7.61}$$

Proof. Direct consequence of Lemma 358 (*open intervals with rational bounds cover open interval*), Lemma 356 (*rational approximation of lower bound of open interval*), and Lemma 357 (*rational approximation of upper bound of open interval*). □

Lemma 362 ***($\overline{\mathbb{R}}$ is second-countable).***
Let $\overline{\mathcal{T}} \stackrel{\text{def.}}{=} \mathcal{T}_{\overline{\mathbb{R}}}(\mathcal{R}^{o}_{\overline{\mathbb{R}}})$ be the order topology on $\overline{\mathbb{R}}$. Then, the open intervals with rational bounds constitute a topological basis of $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$. Hence, $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$ is second-countable.

Proof. Direct consequence of Lemma 310 (*topology of $\overline{\mathbb{R}}$*), Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*, similar proof for $\overline{\mathbb{R}}$), Lemma 361 (*open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$*), Definition 254 (*topological basis*), **countability of $\mathbb{Q}^2$**, **compatibility of union with countability**, and Definition 262 (*second-countability*). □

7.5.4 Infimum, supremum

Lemma 363 ***(extrema of constant function).*** *Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Let $a \in \overline{\mathbb{R}}$. Assume that f is constant of value a. Then,*

$$\sup(f(X)) = \max(f(X)) = \inf(f(X)) = \min(f(X)) = a. \tag{7.62}$$

Proof. Direct consequence of **the definition of constant function**, Lemma 8 (*equivalent definition of maximum*), Definition 7 (*maximum*), Lemma 15 (*equivalent definition of minimum*), and Definition 14 (*minimum*). □

Lemma 364 ***(equivalent definition of finite infimum).*** *Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Let $l \in \mathbb{R}$. Then, we have $l = \inf(f(X))$ iff l is a lower bound of $f(X)$, and there exists $(x_n)_{n\in\mathbb{N}} \in X$ such that $(f(x_n))_{n\in\mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit l.*

Proof. **"Left" implies "right".** Assume first that $l = \inf(f(X))$. Then, from Definition 9 (*infimum*), l is a (finite) lower bound for $f(X)$. Thus, from Lemma 13 (*finite infimum discrete*), for all $n \in \mathbb{N}$, there exists $x'_n \in X$ such that $f(x'_n) < l + \frac{1}{n+1}$. Let $x_0 \stackrel{\text{def.}}{=} x'_0$. For all $n \in \mathbb{N}$, let $x_{n+1} \stackrel{\text{def.}}{=} \arg\min(f(x_n), f(x'_{n+1}))$. Then, for all $n \in \mathbb{N}$, we have

$$f(x_{n+1}) \leqslant f(x_n) \leqslant f(x'_n) < l + \frac{1}{n+1}.$$

Thus, from **the definition of monotone sequence**, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is nonincreasing. Moreover, let $k \in \mathbb{N}$. Let $N \stackrel{\text{def.}}{=} k$. Let $n \in \mathbb{N}$ such that $n \geqslant N$. Then, from **ordered field properties of $\mathbb{R}$**, we have $f(x_n) \leqslant f(x_N) = f(x_k) < l + \frac{1}{k+1}$. Hence, from Lemma 269 (*equivalent definition of convergent sequence*), the sequence $(f(x_n))_{n\in\mathbb{N}}$ is convergent with limit l.

"Right" implies "left". Conversely, assume now that l is a (finite) lower bound of $f(X)$, and that there exists $(x_n)_{n\in\mathbb{N}} \in X$ such that $(f(x_n))_{n\in\mathbb{N}}$ is nonincreasing with limit l. Let $n \in \mathbb{N}$, and let $\varepsilon_n \stackrel{\text{def.}}{=} \frac{1}{2(n+1)} > 0$. Then, from Definition 27 (*convergent sequence*), let $N \in \mathbb{N}$ such that for all $p \geqslant N$, $|f(x_p) - l| < \varepsilon_n$. Let $x'_n \stackrel{\text{def.}}{=} x_N$. Then, from **ordered field properties of $\mathbb{R}$**, we have $f(x'_n) \leqslant l + \varepsilon_n < l + \frac{1}{n+1}$. Hence, from Lemma 13 (*finite infimum discrete*), we have $l = \inf(f(X))$.

Therefore, we have the equivalence. □

Lemma 365 ***(equivalent definition of finite infimum in $\overline{\mathbb{R}}$).*** *Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Let $l \in \mathbb{R}$. Then, we have $l = \inf(f(X))$ iff l is a lower bound of $f(X)$, and there exists $(x_n)_{n\in\mathbb{N}} \in X$ such that $(f(x_n))_{n\in\mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit l.*

Proof. **Case $-\infty \in f(X)$.** Then, from **the definition of $-\infty$**, and Definition 9 (*infimum*, lower bound), $-\infty$ is the only lower bound of $f(X)$. Thus, since $l \in \mathbb{R}$, both propositions "$l = \inf(f(X))$" and "l is a lower bound of $f(X)$" are wrong. Hence, we have the equivalence.

Case $f(X) \subset \mathbb{R}$. Direct consequence of Lemma 364 (*equivalent definition of finite infimum*).

Case $-\infty \notin f(X)$, and $\infty \in f(X)$. Then, from Lemma 363 (*extrema of constant function*, contrapositive with $\inf(f(X)) \neq \infty$), the function cannot be constant of value ∞. Thus, since $-\infty \notin f(X)$, let $\tilde{x} \in X$ such that $f(\tilde{x}) \in \mathbb{R}$. Let $\pi_f : X \to X$ be the projection defined by

$$\pi_f(x) \stackrel{\text{def.}}{=} \begin{cases} x & \text{when } f(x) \neq \infty, \\ \tilde{x} & \text{otherwise.} \end{cases}$$

Let $\tilde{f} \stackrel{\text{def.}}{=} f \circ \pi_f$. Then, by construction, we have $\tilde{f}(X) \subset \mathbb{R}$ and $\tilde{f} \leqslant f$.

Let l be a lower bound of $\tilde{f}(X)$. Then, l is also a lower bound of $f(X)$. Conversely, let l be a lower bound of $f(X)$. Let $x \in X$. Then, from the definitions of π_f and $\tilde{f}$, we have

$$l \leqslant f(\pi_f(x)) = \tilde{f}(x).$$

Thus, l is also a lower bound of $\tilde{f}(X)$. Hence, $f(X)$ and $\tilde{f}(X)$ have the same lower bounds.

Moreover, from Definition 9 (*infimum*, lower bound), $\inf(f(X))$ is also a lower bound of $\tilde{f}(X)$, and $\inf(\tilde{f}(X))$ is also a lower bound of $f(X)$. Thus, from Definition 9 (*infimum*, greatest lower bound), we have $\inf(f(X)) \leqslant \inf(\tilde{f}(X))$, and $\inf(\tilde{f}(X)) \leqslant \inf(f(X))$. Hence, from **totally ordered set properties of $\overline{\mathbb{R}}$**, we have $\inf(\tilde{f}(X)) = \inf(f(X))$.

Furthermore, let $(\tilde{x}_n)_{n\in\mathbb{N}}$ be a sequence in X. Then, from the definition of π_f and $\tilde{f}$, the sequence $(x_n \stackrel{\text{def.}}{=} \pi_f(\tilde{x}_n))_{n\in\mathbb{N}} \in X$ is such that both sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(\tilde{f}(\tilde{x}_n))_{n\in\mathbb{N}}$ are identical. Conversely, assume that $(x_n)_{n\in\mathbb{N}} \in X$ is such that $(f(x_n))_{n\in\mathbb{N}}$ is in $\mathbb{R}$. Then, from the definition of π_f and $\tilde{f}$, both sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(\tilde{f}(x_n))_{n\in\mathbb{N}}$ are identical.

Therefore, from what precedes, we have the equivalences $l = \inf(f(X))$ iff $l = \inf(\tilde{f}(X))$ iff l is a lower bound of $\tilde{f}(X)$, and there exists a sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \in X$ such that $(\tilde{f}(\tilde{x}_n))_{n\in\mathbb{N}}$ is nonincreasing with limit l iff l is a lower bound of $f(X)$, and there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $(f(x_n))_{n\in\mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit l □

Lemma 366 ***(equivalent definition of infimum).*** *Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Let $l \in \overline{\mathbb{R}}$. Then, we have $l = \inf(f(X))$ iff l is a lower bound of $f(X)$, and there exists $(x_n)_{n\in\mathbb{N}} \in X$ such that $(f(x_n))_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit l.*

Proof. **Case $\inf(f(X)) = -\infty$.**

"Left" implies "right". Assume first that $l = \inf(f(X)) = -\infty$. Then, from **the definition of $-\infty$**, l is a lower bound of $f(X)$. Moreover, from Lemma 11 (*finite infimum*, contrapositive), for all $m \in \mathbb{R}$, there exists $x \in X$ such that $f(x) < m$. For all $n \in \mathbb{N}$, let $x'_n \in X$ such that $f(x'_n) < -n$. Let $x_0 \stackrel{\text{def.}}{=} x'_0$. For all $n \in \mathbb{N}$, let $x_{n+1} \stackrel{\text{def.}}{=} \arg\min(f(x_n), f(x'_{n+1}))$. Then, for all $n \in \mathbb{N}$, we have $f(x_{n+1}) \leqslant f(x_n) \leqslant f(x'_n) < -n$. Thus, from **the definition of monotone sequence**, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is nonincreasing. Moreover, let $k \in \mathbb{N}$. Let $N \stackrel{\text{def.}}{=} k$. Let $n \in \mathbb{N}$ such that $n \geqslant N$. Then, from **totally ordered set properties of $\mathbb{R}$**, we have

$$f(x_n) \leqslant f(x_N) = f(x_k) < -k.$$

Hence, from Lemma 314 (*convergence towards $-\infty$*), the sequence $(f(x_n))_{n\in\mathbb{N}}$ is convergent in $\overline{\mathbb{R}}$ with limit $-\infty = l$.

"Right" implies "left". Conversely, assume now that l is a lower bound of $f(X)$. Then, from Definition 9 (*infimum*, greatest lower bound), and **the definition of $-\infty$**, we have

$$-\infty \leqslant l \leqslant \inf(f(X)) = -\infty.$$

Hence, we have $l = -\infty = \inf(f(X))$.

Case $\inf(f(X)) = \infty$.

Then, from Definition 9 (*infimum*, lower bound), and **the definition of ∞**, we have for all $x \in X$, $\infty = \inf(f(X)) \leqslant f(x) \leqslant \infty$. Which means that f is a constant function of value ∞. Thus, from **the definition of ∞**, every extended number is a lower bound of $f(X) = \{\infty\}$. Moreover, for all sequence $(x_n)_{n\in\mathbb{N}} \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is stationary of value ∞. Thus, from **the definition of monotone sequence**, and Lemma 34 (*stationary sequence is convergent*), $(f(x_n))_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit ∞. For instance, the stationary sequence $(\tilde{x})_{n\in\mathbb{N}}$ for some $\tilde{x} \in X \neq \varnothing$ is such a sequence. Hence, we have the equivalence $l = \infty (= \inf(f(X)))$ iff there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that the sequence $(f(x_n))_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit l.

Case $\inf(f(X)) \in \mathbb{R}$.
Direct consequence of Lemma 365 (*equivalent definition of finite infimum in* $\overline{\mathbb{R}}$).

Therefore, we always have the equivalence. □

Lemma 367 ***(infimum is smaller than supremum).***
Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $\inf(f(X)) \leqslant \sup(f(X))$.

Proof. Direct consequence of Definition 9 (*infimum*, lower bound), Definition 2 (*supremum*, upper bound), and Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*, transitivity). □

Lemma 368 ***(infimum is monotone).*** *Let X and Y be nonempty sets. Assume that $Y \subset X$. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $\inf(f(X)) \leqslant \inf(f(Y))$.*

Proof. Direct consequence of Definition 9 (*infimum*, lower bound with X, then greatest lower bound with Y), and Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*, transitivity). □

Lemma 369 ***(supremum is monotone).*** *Let X and Y be nonempty sets. Assume that $Y \subset X$. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $\sup(f(Y)) \leqslant \sup(f(X))$.*

Proof. Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 368 (*infimum is monotone*), and **monotonicity of additive inverse**. □

Lemma 370 ***(compatibility of infimum with absolute value).***
Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $|\inf(f(X))| \leqslant \sup(|f(X)|)$.

Proof. From Lemma 298 (*equivalent definition of absolute value in* $\overline{\mathbb{R}}$, $f \leqslant |f|$), Lemma 368 (*infimum is monotone*), Lemma 367 (*infimum is smaller than supremum*), Lemma 303 (*absolute value in* $\overline{\mathbb{R}}$ *is even*), and Lemma 369 (*supremum is monotone*), we have

$$\inf(f(X)) \leqslant \inf|f(X)| \leqslant \sup|f(X)| \quad \text{and} \quad \sup(-f(X)) \leqslant \sup|f(X)|.$$

Case $\inf(f(X)) \geqslant 0$. Then, from Lemma 298 (*equivalent definition of absolute value in* $\overline{\mathbb{R}}$), we have $|\inf(f(X))| = \inf(f(X)) \leqslant \sup|f(X)|$. **Case $\inf(f(X)) < 0$.** Then, from Lemma 10 (*duality infimum-supremum*), we have $|\inf(f(X)) = -\inf(f(X)) = \sup(-f(X)) \leqslant \sup|f(X)|$. □

Lemma 371 ***(compatibility of supremum with absolute value).***
Let X be a nonempty set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $|\sup(f(X))| \leqslant \sup(|f(X)|)$.

Proof. Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 370 (*compatibility of infimum with absolute value*, with $(-f_n)_{n\in\mathbb{N}}$), and Lemma 303 (*absolute value in* $\overline{\mathbb{R}}$ *is even*). □

Lemma 372 ***(compatibility of translation with infimum).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $p \in \mathbb{N}$, we have $\inf_{n\in\mathbb{N}} f_n \leqslant \inf_{n\in\mathbb{N}} f_{n+p}$.*

Proof. Direct consequence of Lemma 368 (*infimum is monotone*, with $[p..\infty) \subset \mathbb{N}$). □

Lemma 373 ***(compatibility of translation with supremum).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $p \in \mathbb{N}$, we have $\sup_{n\in\mathbb{N}} f_{n+p} \leqslant \sup_{n\in\mathbb{N}} f_n$.*

Proof. Direct consequence of Lemma 369 (*supremum is monotone*, with $[p..\infty) \subset \mathbb{N}$). □

Lemma 374 ***(infimum of sequence is monotone).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that for all $n \in \mathbb{N}$, $f_n \leqslant g_n$ Then, we have $\inf_{n\in\mathbb{N}} f_n \leqslant \inf_{n\in\mathbb{N}} g_n$.

Proof. Direct consequence of Definition 9 (*infimum*, with $X \stackrel{\text{def.}}{=} \mathbb{N}$, lower bound for $\inf_{n\in\mathbb{N}} f_n$ and greatest lower bound for $\inf_{n\in\mathbb{N}} g_n$), and Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*, transitivity). □

Lemma 375 *(supremum of sequence is monotone).*
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that for all $n \in \mathbb{N}$, $f_n \leqslant g_n$ Then, we have $\sup_{n\in\mathbb{N}} f_n \leqslant \sup_{n\in\mathbb{N}} g_n$.

Proof. Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 374 (*infimum of sequence is monotone*, with $-g_n \leqslant -f_n$), and **monotonicity of additive inverse**. □

Lemma 376 *(infimum of bounded sequence is bounded).*
Let X be a nonempty set. Let $a, b \in \overline{\mathbb{R}}$. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$.
If for all $n \in \mathbb{N}$, $f_n \leqslant b$, then we have $\inf_{n\in\mathbb{N}} f_n \leqslant b$.
If for all $n \in \mathbb{N}$, $a \leqslant f_n$, then we have $a \leqslant \inf_{n\in\mathbb{N}} f_n$.

Proof. Direct consequence of Lemma 363 (*extrema of constant function*, with $X \stackrel{\text{def.}}{=} \mathbb{N}$), and Lemma 374 (*infimum of sequence is monotone*). □

Lemma 377 *(supremum of bounded sequence is bounded).*
Let X be a nonempty set. Let $a, b \in \overline{\mathbb{R}}$. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$.
If for all $n \in \mathbb{N}$, $f_n \leqslant b$, then we have $\sup_{n\in\mathbb{N}} f_n \leqslant b$.
If for all $n \in \mathbb{N}$, $a \leqslant f_n$, then we have $a \leqslant \sup_{n\in\mathbb{N}} f_n$.

Proof. Direct consequence of Lemma 363 (*extrema of constant function*, with $X \stackrel{\text{def.}}{=} \mathbb{N}$), and Lemma 375 (*supremum of sequence is monotone*). □

7.5.5 Limit inferior, limit superior

Lemma 378 ***(limit inferior).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $x \in X$, $(\inf_{p\in\mathbb{N}} f_{n+p}(x))_{n\in\mathbb{N}}$ is nondecreasing, and we have*

(7.63)
$$\forall x \in X, \quad \lim_{n\to\infty} \left(\inf_{p\in\mathbb{N}} f_{n+p}(x) \right) = \sup_{n\in\mathbb{N}} \left(\inf_{p\in\mathbb{N}} f_{n+p}(x) \right) \quad \in \overline{\mathbb{R}}.$$

The limit inferior *of the sequence is the function $X \to \overline{\mathbb{R}}$ defined by*

(7.64)
$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) \stackrel{\text{def.}}{=} \lim_{n\to\infty} \left(\inf_{p\in\mathbb{N}} f_{n+p}(x) \right).$$

Proof. Direct consequence of Lemma 368 (*infimum is monotone*), and **completeness of $\overline{\mathbb{R}}$ (a nondecreasing sequence is convergent and its limit is its least upper bound)**. □

Lemma 379 ***(limit inferior is ∞).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Let $x \in X$. Assume that $\liminf_{n\to\infty} f_n(x) = \infty$. Then, we have $\lim_{n\to\infty} f_n(x) = \infty$*

Proof. Let $a \in \mathbb{R}$. Then, from **the definition of ∞**, we have $a < \liminf_{n\to\infty} f_n(x)$. Thus, from Lemma 378 (*limit inferior*), and **the definition of the limit**, there exists $N \in \mathbb{N}$ such that $a < \inf_{p\in\mathbb{N}} f_{N+p}(x)$. Hence, from Definition 9 (*infimum*, lower bound), we have for all $p \in \mathbb{N}$, $a < \inf_{p\in\mathbb{N}} f_{N+p}(x) \leqslant f_{N+p}(x)$. Therefore, from **the definition of the limit**, we have $\lim_{n\to\infty} f_n(x) = \infty$. □

Lemma 380 ***(equivalent definition of the limit inferior).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $x \in X$, $\liminf_{n\to\infty} f_n(x)$ is the smallest cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.*

Proof. Let $x \in X$. For all $n \in \mathbb{N}$, let $F_n^-(x) \stackrel{\text{def.}}{=} \inf_{p\in\mathbb{N}} f_{n+p}(x)$.
Let $\underline{f}(x) \stackrel{\text{def.}}{=} \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} F_n^-(x) \in \overline{\mathbb{R}}$.

Let us first show that $\underline{f}(x)$ is a cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Case $\underline{f}(x)$ is finite. Let $\varepsilon > 0$. Let $M \in \mathbb{N}$. Then, from Lemma 378 (*limit inferior*, $(F_n^-(x))_{n\in\mathbb{N}}$ is nondecreasing), and Definition 27 (*convergent sequence*, with $(F_n^-(x))_{n\in\mathbb{N}}$), there exists $N \in \mathbb{N}$ such that we have

$$\forall k \in [N..\infty), \quad \underline{f}(x) - \varepsilon \leqslant F_k^-(x) \leqslant \underline{f}(x).$$

Let $K \stackrel{\text{def.}}{=} \max(M, N)$. Then, from Lemma 11 (*finite infimum*, for F_K^-), there exists $k \geqslant K \geqslant M$ such that

$$\underline{f}(x) - \varepsilon \leqslant F_K^-(x) \leqslant f_k(x) < F_K^-(x) + \varepsilon \leqslant \underline{f}(x) + \varepsilon.$$

Hence, from Definition 271 (*cluster point*), $\underline{f}(x)$ is a cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Case $\underline{f}(x) = -\infty$. From Lemma 378 (*limit inferior*), and Definition 2 (*supremum*, upper bound), we have for all $n \in \mathbb{N}$, $F_n^-(x) = -\infty$. Let $n \in \mathbb{N}$. Let $a \in \mathbb{R}$. Then, $F_n^-(x) < a$. Thus, from Definition 9 (*infimum*, greatest lower bound, contrapositive), there exists $P \in \mathbb{N}$ such that $f_{n+P}(x) < a$. Hence, from **the definition of cluster point in $-\infty$**, $\underline{f}(x) = -\infty$ is a cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Case $\underline{f}(x) = \infty$. Then, from Lemma 379 (*limit inferior is ∞*), we have $\lim_{n\to\infty} f_n(x) = \infty$, and from **the fact that the limit of a convergent sequence is its only cluster point**, $\underline{f}(x) = \infty$ is a cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Now, let $f(x)$ be a cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$. Let us show that $\underline{f}(x) \leqslant f(x)$.

Case $f(x) = \infty$. Then, from **the definition of ∞**, we have $\underline{f}(x) \leqslant f(x)$.

Case $f(x) < \infty$. Let $a \in \mathbb{R}$ such that $f(x) < a$. Let $\varepsilon \overset{\text{def.}}{=} \frac{a-f(x)}{2} > 0$. Let $n \in \mathbb{N}$. Then, from Definition 271 (*cluster point*), there exists $k \geqslant n$ such that $f_k(x) \leqslant f(x) + \varepsilon < a$. Thus, from Definition 9 (*infimum*, lower bound), $F_n^-(x) < a$. Hence, from **monotonicity of the limit**, and Lemma 378 (*limit inferior*), we have $\underline{f}(x) = \lim_{n\to\infty} F_n^-(x) \leqslant a$. Since this is true for all $a > f(x)$, we also have $\underline{f}(x) \leqslant f(x)$.

Therefore, for all $x \in X$, $\underline{f}(x)$ is the smallest cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$. □

Lemma 381 ***(limit inferior is invariant by translation).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, we have

$$\forall k \in \mathbb{N}, \quad \forall x \in X, \quad \liminf_{n\to\infty} f_{k+n}(x) = \liminf_{n\to\infty} f_n(x). \tag{7.65}$$

Proof. Direct consequence of Lemma 380 (*equivalent definition of the limit inferior*), Definition 271 (*cluster point*), and **compatibility of translation with limit**. □

Lemma 382 ***(limit inferior is monotone).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that $(f_n)_{n\in\mathbb{N}} \leqslant (g_n)_{n\in\mathbb{N}}$ from some rank:*

$$\exists N \in \mathbb{N}, \ \forall n \in [N..\infty), \ \forall x \in X, \quad f_n(x) \leqslant g_n(x). \tag{7.66}$$

Then, we have

$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) \leqslant \liminf_{n\to\infty} g_n(x). \tag{7.67}$$

Proof. Direct consequence of Lemma 381 (*limit inferior is invariant by translation*, with $k \overset{\text{def.}}{=} N$), Lemma 378 (*limit inferior*), Lemma 374 (*infimum of sequence is monotone*), and Lemma 375 (*supremum of sequence is monotone*). □

Lemma 383 ***(limit superior).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $x \in X$, $(\sup_{p\in\mathbb{N}} f_{n+p}(x))_{n\in\mathbb{N}}$ is nonincreasing, and we have*

$$\forall x \in X, \quad \lim_{n\to\infty} \left(\sup_{p\in\mathbb{N}} f_{n+p}(x) \right) = \inf_{n\in\mathbb{N}} \left(\sup_{p\in\mathbb{N}} f_{n+p}(x) \right) \quad \text{in } \overline{\mathbb{R}}. \tag{7.68}$$

The limit superior *of the sequence is the function $X \to \overline{\mathbb{R}}$ defined by*

$$\forall x \in X, \quad \limsup_{n\to\infty} f_n(x) \overset{\text{def.}}{=} \lim_{n\in\mathbb{N}} \left(\sup_{p\in\mathbb{N}} f_{n+p}(x) \right). \tag{7.69}$$

Proof. Let $x \in X$. For all $n \in \mathbb{N}$, let $F_n(x) \overset{\text{def.}}{=} \{f_{n+p}(x) \,|\, p \in \mathbb{N}\}$. Then, we have

$$\forall n \in \mathbb{N}, \quad \sup F_n(x) = \sup_{p\in\mathbb{N}} f_{n+p}(x) \quad \wedge \quad F_n(x) \subset F_{n+1}(x) \subset \overline{\mathbb{R}}.$$

Thus, the sequence $(\sup F_n(x))_{n\in\mathbb{N}}$ is nonincreasing in $\overline{\mathbb{R}}$. Therefore, from **completeness of $\overline{\mathbb{R}}$**, the sequence is convergent and its limit is its greatest lower bound $\inf_{n\in\mathbb{N}}(\sup F_n(x))$. □

Lemma 384 ***(duality limit inferior-limit superior).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, we have

$$\forall x \in X, \quad \limsup_{n\to\infty} f_n(x) = -\liminf_{n\to\infty}(-f_n(x)). \tag{7.70}$$

Proof. Direct consequence of Lemma 383 (*limit superior*), Lemma 378 (*limit inferior*), **linearity and uniqueness of the limit**, and Lemma 10 (*duality infimum-supremum*, with $X \overset{\text{def.}}{=} \mathbb{N}$). □

Lemma 385 ***(equivalent definition of limit superior).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, for all $x \in X$, $\limsup_{n\to\infty} f_n(x)$ is the largest cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Proof. Let $x \in X$. Let $\overline{f}(x) \overset{\text{def.}}{=} \limsup_{n\to\infty} f_n(x)$. Then, from Lemma 384 (*duality limit inferior-limit superior*), and Lemma 380 (*equivalent definition of the limit inferior*), $-\overline{f}(x)$ is the smallest cluster point of the sequence $(-f_n(x))_{n\in\mathbb{N}}$. Therefore, from **linearity of the limit**, and **totally ordered set properties of $\overline{\mathbb{R}}$**, $\overline{f}(x)$ is the largest cluster point of the sequence $(f_n(x))_{n\in\mathbb{N}}$. □

Lemma 386 ***(limit inferior is smaller than limit superior).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Then, we have

$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) \leqslant \limsup_{n\to\infty} f_n(x). \tag{7.71}$$

Proof. Direct consequence of Lemma 380 (*equivalent definition of the limit inferior*), Lemma 385 (*equivalent definition of limit superior*), and Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*)transitivity. □

Lemma 387 ***(limit superior is monotone).*** *Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that $(f_n)_{n\in\mathbb{N}} \leqslant (g_n)_{n\in\mathbb{N}}$ from some rank:*

$$\exists N \in \mathbb{N},\ \forall n \in [N..\infty),\ \forall x \in X, \quad f_n(x) \leqslant g_n(x). \tag{7.72}$$

Then, we have

$$\forall x \in X, \quad \limsup_{n\to\infty} f_n(x) \leqslant \limsup_{n\to\infty} g_n(x). \tag{7.73}$$

Proof. Direct consequence of **monotonicity of additive inverse in $\overline{\mathbb{R}}$**, Lemma 382 (*limit inferior is monotone*, with $-g_n \leqslant -f_n$), and Lemma 384 (*duality limit inferior-limit superior*). □

Lemma 388 ***(compatibility of limit inferior with absolute value).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}_+$. Then, we have

$$\forall x \in X, \quad \left|\liminf_{n\to\infty} f_n(x)\right| \leqslant \limsup_{n\to\infty} |f_n(x)|. \tag{7.74}$$

Proof. From Lemma 370 (*compatibility of infimum with absolute value*, with $X \overset{\text{def.}}{=} [n..\infty) \times X$), for all $x \in X$, for all $n \in \mathbb{N}$, we have

$$\left|\inf_{p\in\mathbb{N}} f_{n+p}(x)\right| \leqslant \sup_{p\in\mathbb{N}} |f_{n+p}(x)|.$$

Therefore, from Lemma 378 (*limit inferior*), **compatibility of the limit with the absolute value**, **monotonicity of the limit (when $n \to \infty$)**, and Lemma 383 (*limit superior*), we have

$$\left|\liminf_{n\to\infty} f_n(x)\right| = \left|\lim_{n\to\infty} \inf_{p\in\mathbb{N}} f_{n+p}(x)\right| = \lim_{n\to\infty} \left|\inf_{p\in\mathbb{N}} f_{n+p}(x)\right| \leqslant \lim_{n\to\infty} \sup_{p\in\mathbb{N}} |f_{n+p}(x)| = \limsup_{n\to\infty} |f_n(x)|.$$

□

Lemma 389 ***(compatibility of limit superior with absolute value).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}_+$. Then, we have

$$\forall x \in X, \quad \left|\limsup_{n\to\infty} f_n(x)\right| \leqslant \limsup_{n\to\infty} |f_n(x)|. \tag{7.75}$$

Proof. Direct consequence of Lemma 384 (*duality limit inferior-limit superior*), Lemma 388 (*compatibility of limit inferior with absolute value*, with $-f_n$), and Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*). □

Definition 390 *(pointwise convergence).*
Let X be a nonempty set. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $X \to \overline{\mathbb{R}}$ is said *pointwise convergent* iff for all $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is convergent in $\overline{\mathbb{R}}$.

Lemma 391 *(limit inferior and limit superior of pointwise convergent).*
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that the sequence is pointwise convergent. Then, we have

$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x). \tag{7.76}$$

Proof. Direct consequence of Definition 390 (*pointwise convergence*), Lemma 380 (*equivalent definition of the limit inferior*), Lemma 385 (*equivalent definition of limit superior*), and **the fact that a convergent sequence has its limit as the unique cluster point**. □

Lemma 392 *(limit inferior bounded from below).*
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Let $m \in \overline{\mathbb{R}}$. Assume that the sequence is bounded by m from below from some rank:

$$\exists N \in \mathbb{N}, \ \forall n \in [N..\infty), \ \forall x \in X, \quad m \leqslant f_n(x). \tag{7.77}$$

Then, we have

$$\forall x \in X, \quad m \leqslant \liminf_{n\to\infty} f_n(x). \tag{7.78}$$

Proof. Direct consequence of Lemma 382 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence*, constant sequence is stationary ($N = 0$)), and Lemma 391 (*limit inferior and limit superior of pointwise convergent*, with the constant sequence of value m). □

Lemma 393 *(limit inferior bounded from above).*
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Let $M \in \overline{\mathbb{R}}$. Assume that the sequence is bounded by M from above from some rank:

$$\exists N \in \mathbb{N}, \ \forall n \in [N..\infty), \ \forall x \in X, \quad f_n(x) \leqslant M. \tag{7.79}$$

Then, we have

$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) \leqslant M. \tag{7.80}$$

Proof. Direct consequence of Lemma 382 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence*, constant sequence is stationary ($N = 0$)), and Lemma 391 (*limit inferior and limit superior of pointwise convergent*, with the constant sequence of value M). □

Lemma 394 *(limit superior bounded from below).*
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Let $m \in \overline{\mathbb{R}}$. Assume that the sequence is bounded by m from below from some rank:

$$\exists N \in \mathbb{N}, \ \forall n \in [N..\infty), \ \forall x \in X, \quad m \leqslant f_n(x). \tag{7.81}$$

Then, we have

$$\forall x \in X, \quad m \leqslant \limsup_{n\to\infty} f_n(x). \tag{7.82}$$

Proof. Direct consequence of Lemma 393 (*limit inferior bounded from above*, with $f_n \overset{\text{def.}}{=} -f_n$ and $M \overset{\text{def.}}{=} -m$), and Lemma 384 (*duality limit inferior-limit superior*). □

Lemma 395 ***(limit superior bounded from above).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Let $M \in \overline{\mathbb{R}}$. Assume that the sequence is bounded by M from above from some rank:

$$\exists N \in \mathbb{N},\ \forall n \in [N..\infty),\ \forall x \in X, \quad f_n(x) \leqslant M. \tag{7.83}$$

Then, we have

$$\forall x \in X, \quad \limsup_{n\to\infty} f_n(x) \leqslant M. \tag{7.84}$$

Proof. Direct consequence of Lemma 392 (*limit inferior bounded from below*, with $f_n \overset{\text{def.}}{=} -f_n$ and $m \overset{\text{def.}}{=} -M$), and Lemma 384 (*duality limit inferior-limit superior*). □

Lemma 396 ***(limit inferior, limit superior and pointwise convergence).***
Let X be a nonempty set. Let $(f_n)_{n\in\mathbb{N}} : X \to \overline{\mathbb{R}}$. Assume that

$$\forall x \in X, \quad \limsup_{n\to\infty} f_n(x) \leqslant \liminf_{n\to\infty} f_n(x). \tag{7.85}$$

Then, the sequence is pointwise convergent and

$$\forall x \in X, \quad \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x). \tag{7.86}$$

Proof. Let $x \in X$. Then, from Lemma 386 (*limit inferior is smaller than limit superior*), we have $\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) \overset{\text{def.}}{=} f(x) \in \overline{\mathbb{R}}$.

Case $f(x) = \infty$. Then, from Lemma 379 (*limit inferior is ∞*), we have

$$\lim_{n\to\infty} f_n(x) = \infty = f(x).$$

Case $f(x) = -\infty$. Then, from Lemma 384 (*duality limit inferior-limit superior*), and Lemma 379 (*limit inferior is ∞*), we have $\lim_{n\to\infty} f_n(x) = -\infty = f(x)$.

Case $f(x) \in \mathbb{R}$. Let $\varepsilon > 0$. Then, we have

$$f(x) - \varepsilon < \liminf_{n\to\infty} f_n(x) = f(x) = \limsup_{n\to\infty} f_n(x) < f(x) + \varepsilon.$$

Thus, from Lemma 380 (*equivalent definition of the limit inferior*), there exists $N^- \in \mathbb{N}$ such that for all $n \geqslant N^-$, we have $f(x) - \varepsilon < f_n(x)$, and from Lemma 385 (*equivalent definition of limit superior*), there exists $N^+ \in \mathbb{N}$ such that for all $n \geqslant N^+$, we have $f_n(x) < f(x) + \varepsilon$.
Let $N \overset{\text{def.}}{=} \max(N^-, N^+)$. Then, for all $n \geqslant N$, we have $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$. Therefore, from Definition 27 (*convergent sequence*), the sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent with limit $f(x)$. □

7.5.6 Truncating a function

Definition 397 ***(finite part).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. The *finite part of* f is the function $f\mathbb{1}_{f^{-1}(\mathbb{R})}$.

Lemma 398 ***(finite part is finite).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. Then, the finite part of f is finite.

Proof. Direct consequence of Definition 397 (*finite part*), Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \{\pm\infty\} \uplus \mathbb{R}$), **properties of inverse image ($X = f^{-1}(\overline{\mathbb{R}}) = f^{-1}(\pm\infty) \uplus f^{-1}(\mathbb{R})$)**, and **the definition of the indicator function**. □

Definition 399 ***(nonnegative and nonpositive parts).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. The functions $f^+ \stackrel{\text{def.}}{=} \max(f, 0)$ and $f^- \stackrel{\text{def.}}{=} \max(-f, 0)$ are respectively called the *nonnegative part of* f and the *nonpositive part of* f.

Lemma 400 ***(equivalent definition of nonnegative and nonpositive parts).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $f^+ = f\mathbb{1}_{f^{-1}(\overline{\mathbb{R}}_+)}$ and $f^- = -f\mathbb{1}_{f^{-1}(\overline{\mathbb{R}}_-)}$.

Proof. Direct consequence of Definition 399 (*nonnegative and nonpositive parts*), **the definition of the maximum**, and **the definition of the indicator function**. □

Lemma 401 ***(nonnegative and nonpositive parts are nonnegative).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $f^+, f^- \geqslant 0$.

Proof. Direct consequence of Definition 399 (*nonnegative and nonpositive parts*), and **the definition of the maximum**. □

Lemma 402 ***(nonnegative and nonpositive parts are orthogonal).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. Let $x \in X$. Then, we have

$$f^+(x) = 0 \ (\textit{and } f^-(x) = -f(x)) \quad \vee \quad f^-(x) = 0 \ (\textit{and } f^+(x) = f(x)). \tag{7.87}$$

Proof. Direct consequence of **the definition of the maximum**, **the partition of extended real numbers into negative, zero, and positive extended numbers**. □

Lemma 403 ***(decomposition into nonnegative and nonpositive parts).***
Let X be a set. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Proof. Direct consequence of Lemma 402 (*nonnegative and nonpositive parts are orthogonal*, f^+ and f^- cannot take value ∞ at the same point), Definition 282 (*addition in* $\overline{\mathbb{R}}$), and Definition 297 (*absolute value in* $\overline{\mathbb{R}}$). □

Lemma 404 ***(compatibility of nonpositive and nonnegative parts with addition).***
Let X be a set. Let $f, g : X \to \overline{\mathbb{R}}$ such that their sum is well-defined. Then, we have

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+. \tag{7.88}$$

Proof. Let $x \in X$.

Case $f(x)$ and $g(x)$ finite. From Lemma 403 (*decomposition into nonnegative and nonpositive parts*, with $f+g$, then f and g), we have

$$(f+g)^+(x) - (f+g)^-(x) = f(x) + g(x) = (f^+(x) - f^-(x)) + (g^+(x) - g^-(x)).$$

Then, from Definition 399 (*nonnegative and nonpositive parts*), $(f+g)^+(x)$, $(f+g)^-(x)$, $f^+(x)$, $f^-(x)$, $g^+(x)$ and $g^-(x)$, are finite. Hence, from **abelian group properties of $\mathbb{R}$**, we have

$$(f+g)^+(x) + f^-(x) + g^-(x) = (f+g)^-(x) + f^+(x) + g^+(x).$$

Case $f(x)$ or $g(x)$ is ∞. Then, from Definition 282 (*addition in $\overline{\mathbb{R}}$*, rule 3 cannot occur), and Lemma 402 (*nonnegative and nonpositive parts are orthogonal*), we have

$$(f+g)^+(x) = f(x) + g(x) = \infty \quad \text{and} \quad (f^+(x) = \infty \quad \vee \quad g^+(x) = \infty).$$

Thus, from Lemma 401 (*nonnegative and nonpositive parts are nonnegative*), Lemma 321 (*infinity-sum property in $\overline{\mathbb{R}}_+$*), Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*), and Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*), we have

$$(f+g)^+(x) + f^-(x) + g^-(x) = \infty \quad \text{and} \quad (f+g)^-(x) + f^+(x) + g^+(x) = \infty.$$

Hence, Equation (7.88) is satisfied.

Case $f(x)$ or $g(x)$ is $-\infty$. Same reasoning with the nonpositive parts.

Therefore, we always have the identity. □

Lemma 405 ***(compatibility of nonpositive and nonnegative parts with mask).*** *Let X be a set. Let $A \subset X$. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $(f\mathbb{1}_A)^\pm = f^\pm \mathbb{1}_A$*

Proof. Direct consequence of Definition 399 (*nonnegative and nonpositive parts*), and **nonnegativeness of the indicator function**. □

Lemma 406 ***(compatibility of nonpositive and nonnegative parts with restriction).*** *Let X be a set. Let $A \subset X$. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $(f_{|A})^\pm = {f^\pm}_{|A}$.*

Proof. Direct consequence of Definition 399 (*nonnegative and nonpositive parts*), and **compatibility of min/max with restriction of function**. □

Chapter 8

Subset systems

Contents

Remark 407. Subset systems of a set X are subsets of its power set $\mathcal{P}(X)$.

Interesting subset systems are those closed under a family of set operations such as complement, union or intersection. The simplest one is π-system that is nonempty and only closed under intersection. The most elaborate one is σ-algebra that is somehow closed under all set operations, including countable union and intersection.

Measure theory and Lebesgue integration are usually based on σ-algebra that is the most general subset system concept with which we can build desirable properties such as σ-additivity for measures, and linearity and powerful convergence theorems for the integral. Thus, the last two sections of this chapter are dedicated to ways to extend weaker subset systems, such as set algebra or λ-system, into a σ-algebra.

8.1 Basic properties

Lemma 408 ***(nonempty and with empty or full).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. If $\mathcal{S}$ contains either the empty set or the full set, then it is nonempty.*
Assume that $\mathcal{S}$ is nonempty and closed under complement. Then, it contains the empty set if it is closed under intersection, and it contains the full set if it is closed under union.

Proof. Direct consequence of **the identities $A \setminus A = \varnothing$, and $A \cup A^c = X$**. □

Lemma 409 ***(with empty and full).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it contains the empty set iff it contains the full set.

Proof. Direct consequence of **the identities $\varnothing^c = X$, and $X^c = \varnothing$**. □

Remark 410.
The local complement is the set difference when the first operand contains the second one.

Lemma 411 ***(closedness under local complement and complement).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set and is closed under local complement. Then, it is closed under complement.*

Proof. Direct consequence of **the identity $A^c = X \setminus A$**. □

Lemma 412 ***(closedness under disjoint union and local complement).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement. Then, it is closed under disjoint union iff it is closed under local complement.*

Proof. Direct consequence of **the identity $A \setminus B = (A^c \cup B)^c$**, **monotonicity of complement (then $A \cap B = \varnothing \Leftrightarrow B \subset A^c$)**, and **the identity $A \cup B = (A^c \setminus B)^c$**. □

Lemma 413 ***(closedness under set difference and local complement).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.*
If $\mathcal{S}$ is closed under set difference, then it is closed under local complement.
If $\mathcal{S}$ is closed under intersection and local complement, then it is closed under set difference.

Proof. Direct consequence of **the identity $A \setminus B = A \setminus (A \cap B)$ with $A \cap B \subset A$**. □

Lemma 414 ***(closedness under intersection and set difference).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.*
If $\mathcal{S}$ is closed under complement and intersection, then it is closed under set difference.
If $\mathcal{S}$ is closed under set difference, then it is closed under intersection.

Proof. Direct consequence of **the identity $A \cap B = A \setminus (A \setminus B)$**. □

Lemma 415 ***(closedness under union and intersection).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement. Then, it is closed under union iff it is closed under intersection.*

Proof. Direct consequence of **De Morgan's laws**. □

Lemma 416 ***(closedness under union and set difference).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement. Then, it is closed under union iff it is closed under set difference.*

Proof. Direct consequence of Lemma 415 (*closedness under union and intersection*), and Lemma 414 (*closedness under intersection and set difference*). □

Lemma 417 ***(closedness under finite operations).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.*
$\mathcal{S}$ is closed under finite intersection iff it is closed under intersection.
$\mathcal{S}$ is closed under finite union iff it is closed under union.
$\mathcal{S}$ is closed under finite disjoint union iff it is closed under disjoint union.

Proof. Direct consequence of **induction on the number of operands**. □

Lemma 418 ***(closedness under finite union and intersection).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement. Then, it is closed under finite union iff it is closed under finite intersection, and it is closed under finite monotone union iff it is closed under finite monotone intersection.*

Proof. Direct consequence of Lemma 417 (*closedness under finite operations*), Lemma 415 (*closedness under union and intersection*), **monotonicity of complement**, and **De Morgan's laws**. □

Remark 419.
Note that obviously, closedness under (finite or countable) union implies closedness under (finite or countable) disjoint union. In the same way, closedness under finite or countable intersection (resp. union) implies closedness under finite or countable monotone intersection (resp. union).

Remark 420. Note that closedness under a countable subset operation actually means that the subset system is closed under this operation with at most a countable number of operands, i.e. it is also valid for a finite number (e.g. two). Except for countable disjoint union (see next lemma).

Lemma 421 ***(closedness under countable and finite disjoint union).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the empty set and is closed under countable disjoint union. Then, it is closed under finite disjoint union.

Proof. Direct consequence of **extension of finite disjoint union into countable disjoint union using the empty set**. □

Lemma 422 ***(closedness under countable disjoint union and local complement).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set and is closed under complement and countable disjoint union. Then, it is closed under local complement.

Proof. Direct consequence of Lemma 409 (*with empty and full*, $\varnothing \in \mathcal{S}$), Lemma 421 (*closedness under countable and finite disjoint union*, $\mathcal{S}$ closed under finite disjoint union), Lemma 417 (*closedness under finite operations*, $\mathcal{S}$ closed under disjoint union), and Lemma 412 (*closedness under disjoint union and local complement*). □

Lemma 423 ***(closedness under countable union and intersection).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it is closed under countable union iff it is closed under countable intersection, and it is closed under countable monotone union iff it is closed under countable monotone intersection.

Proof. Direct consequence of **De Morgan's laws**, and **monotonicity of complement**. □

Lemma 424 ***(closedness under countable disjoint and monotone union).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement and countable disjoint union. Then, it is closed under countable monotone union.

Proof. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}$. Assume that for all $n \in \mathbb{N}$, $A_n \subset A_{n+1}$.
Let $B_0 \stackrel{\text{def.}}{=} A_0$, and for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{p\in[0..n]} B_p$. Then, from Lemma 215 (*partition of countable union*), the sequence $(B_n)_{n\in\mathbb{N}}$ is pairwise disjoint, for all $n \in \mathbb{N}$, we have $A_n = \bigcup_{i\in[0..n]} A_i = \biguplus_{i\in[0..n]} B_i$, and $\bigcup_{n\in\mathbb{N}} A_n = \biguplus_{n\in\mathbb{N}} B_n$. Moreover, $B_0 = A_0 \in \mathcal{S}$, and for all $n \in \mathbb{N}$, we have $B_{n+1} = A_{n+1} \setminus A_n \in \mathcal{S}$. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$.
Therefore, $\mathcal{S}$ is closed under countable monotone union. □

Lemma 425 ***(closedness under countable monotone and disjoint union).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement, local complement and countable monotone union. Then, it is closed under countable disjoint union.

Proof. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}$. Assume that for all $p, q \in \mathbb{N}$, $p \neq q$ implies $A_p \cap A_q = \varnothing$.
For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} \biguplus_{p\in[0..n]} A_p$. Then, from **properties of union**, the sequence $(B_n)_{n\in\mathbb{N}}$ is nondecreasing, for all $n \in \mathbb{N}$, we have $\biguplus_{p\in[0..n]} A_p = \bigcup_{p\in[0..n]} B_p = B_n$, and $\biguplus_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$. Moreover, $B_0 = A_0 \in \mathcal{S}$, and from **De Morgan's laws**, and **the definition of set difference**, for all $n \in \mathbb{N}$, we have $B_n \cap A_{n+1} = \biguplus_{p\in[0..n]} A_p \cap A_{n+1} = \varnothing$ (i.e. $B_n \subset A_{n+1}^c$), and

$$B_{n+1} = A_{n+1} \uplus \biguplus_{p\in[0..n]} A_p = A_{n+1} \uplus B_n = (A_{n+1}^c \cap B_n^c)^c = (A_{n+1}^c \setminus B_n)^c.$$

Then, from a trivial induction, for all $n \in \mathbb{N}$, we have $B_n \in \mathcal{S}$. Hence, $\biguplus_{n\in\mathbb{N}} A_n \in \mathcal{S}$.
Therefore, $\mathcal{S}$ is closed under countable disjoint union. □

Lemma 426 ***(closedness under countable disjoint union and countable union).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement, intersection and countable disjoint union. Then, it is closed under countable union.*

Proof. From Lemma 415 (*closedness under union and intersection*), and Lemma 417 (*closedness under finite operations*), $\mathcal{S}$ is closed under finite union.

Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}$. Let $B_0 \stackrel{\text{def.}}{=} A_0$, and for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{p\in[0..n]} B_p$. Then, from Lemma 215 (*partition of countable union*), the sequence $(B_n)_{n\in\mathbb{N}}$ is pairwise disjoint, for all $n \in \mathbb{N}$, we have $\bigcup_{p\in[0..n]} A_p = \biguplus_{p\in[0..n]} B_p$, and $\bigcup_{n\in\mathbb{N}} A_n = \biguplus_{n\in\mathbb{N}} B_n$. Moreover, $B_0 \in \mathcal{S}$, and from Lemma 414 (*closedness under intersection and set difference*), for all $n \in \mathbb{N}$, we have $B_{n+1} = A_{n+1} \setminus \bigcup_{p\in[0..n]} A_p \in \mathcal{S}$. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$.

Therefore, $\mathcal{S}$ is closed under countable union. □

Lemma 427 ***(closedness under countable monotone union and countable union).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement, union and countable monotone union. Then, it is closed under countable union.*

Proof. Direct consequence of Lemma 415 (*closedness under union and intersection*, $\mathcal{S}$ is closed under intersection), Lemma 414 (*closedness under intersection and set difference*, $\mathcal{S}$ is closed under set difference), Lemma 413 (*closedness under set difference and local complement*, $\mathcal{S}$ is closed under local set difference), Lemma 425 (*closedness under countable monotone and disjoint union*, $\mathcal{S}$ is closed under disjoint union), and Lemma 426 (*closedness under countable disjoint union and countable union*). □

8.2 Pi-system

Definition 428 ***(π-system).*** Let X be a set.
A subset Π of $\mathcal{P}(X)$ is called *π-system on X* iff it is nonempty and closed under finite intersection:

$$\Pi \neq \varnothing, \tag{8.1}$$

$$\forall n \in \mathbb{N},\ \forall (A_p)_{p\in[0..n]} \in \Pi,\quad \bigcap_{p\in[0..n]} A_p \in \Pi. \tag{8.2}$$

Remark 429. Note that adding the empty set to a π-system keeps it a π-system. We may replace in the previous definition the nonempty condition (8.1) by the "contain-the-empty-set" condition, and still maintain the Dynkin π–λ theorem (see Theorem 508).

Remark 430. The following set of four lemmas and a definition is written in an almost identical manner for π-systems (statements 431–435), set algebras (441–445), monotone classes (449–453), λ-systems (462–466), and σ-algebras (481–485). They all derive from the definition and properties of intersection, and reflexivity of inclusion, thus allowing for generated systems that are minimum, satisfy monotonicity and idempotence. The very short proofs are almost identical.

Note that, unlike subsequent subset systems, the π-systems require nonemptiness, that shows up in all statements.

Lemma 431 ***(intersection of π-systems).*** *Let X and I be sets. Let $(\Pi_i)_{i\in I}$ be π-systems on X. Then, $\bigcap_{i\in I} \Pi_i$ is closed under intersection, i.e. it is a π-system iff it is nonempty.*

Proof. Direct consequence of Definition 428 (*π-system*), and **the definition of intersection**. □

Definition 432 ***(generated π-system).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. The *π-system generated by G* is the intersection of all π-systems on X containing G; it is denoted $\Pi_X(G)$.

Lemma 433 ***(generated π-system is minimum).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, $\Pi_X(G)$ is the smallest π-system on X containing G.*

Proof. Direct consequence of Definition 432 (*generated π-system*), Lemma 431 (*intersection of π-systems*), and **properties of the intersection**. □

Lemma 434 ***(π-system generation is monotone).*** *Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $\varnothing \neq G_1 \subset G_2$. Then, we have $\Pi_X(G_1) \subset \Pi_X(G_2)$.*

Proof. Direct consequence of **the definition of inclusion ($G_2 \neq \varnothing$)**, and Lemma 433 (*generated π-system is minimum*, with $G \overset{\text{def.}}{=} G_2$, then $G \overset{\text{def.}}{=} G_1$). □

Lemma 435 ***(π-system generation is idempotent).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S} \neq \varnothing$. Then, $\mathcal{S}$ is a π-system on X iff $\Pi_X(\mathcal{S}) = \mathcal{S}$.

Proof. Direct consequence of **reflexivity of inclusion**, and Lemma 433 (*generated π-system is minimum*, $\mathcal{S}$ and $\Pi_X(\mathcal{S})$ are both π-systems containing $\mathcal{S}$). □

8.3 Set algebra

Remark 436. The following concept of "set algebra" is not to be confused with the algebraic structure "algebra over a field" defined in Section 7.2.2.

Note that the same concept is sometimes called "field (of sets)", which is of course not to be confused with the algebraic structure either.

Definition 437 *(set algebra).* Let X be a set. A subset $\mathcal{A}$ of $\mathcal{P}(X)$ is called *set algebra on* X iff it contains the empty set, it is closed under complement, and under finite union:

$$\varnothing \in \mathcal{A}, \tag{8.3}$$

$$\forall A \in \mathcal{A}, \quad A^c \in \mathcal{A}, \tag{8.4}$$

$$\forall n \in \mathbb{N},\ \forall (A_i)_{i\in[0..n]} \in \mathcal{A}, \quad \bigcup_{i\in[0..n]} A_i \in \mathcal{A}. \tag{8.5}$$

Lemma 438 *(equivalent definition of set algebra).*
Let X be a set. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then, $\mathcal{A}$ is a set algebra on X iff (8.4) holds and

$$\varnothing \in \mathcal{A} \quad \vee \quad X \in \mathcal{A} \quad \vee \quad \mathcal{A} \neq \varnothing, \tag{8.6}$$

$$\forall A, B \in \mathcal{A},\ A \cup B \in \mathcal{A} \quad \vee \quad \forall A, B \in \mathcal{A},\ A \cap B \in \mathcal{A}. \tag{8.7}$$

Proof. Direct consequence of Definition 437 (*set algebra*), Lemma 408 (*nonempty and with empty or full*), Lemma 409 (*with empty and full*), Lemma 417 (*closedness under finite operations*), and Lemma 418 (*closedness under finite union and intersection*). □

Lemma 439 *(other equivalent definition of set algebra).* *Let X be a set. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then, $\mathcal{A}$ is a set algebra on X iff it contains the full set X, and it is closed under set difference:*

$$X \in \mathcal{A}. \tag{8.8}$$

$$\forall A, B \in \mathcal{A}, \quad A \setminus B \in \mathcal{A}. \tag{8.9}$$

Proof. Direct consequence of Lemma 438 (*equivalent definition of set algebra*), Lemma 414 (*closedness under intersection and set difference*, both ways), Lemma 413 (*closedness under set difference and local complement*), and Lemma 411 (*closedness under local complement and complement*). □

Lemma 440 *(set algebra is closed under local complement).*
Let X be a set. Let $\mathcal{A}$ be a set algebra on X. Then, $\mathcal{A}$ is closed under local complement.

Proof. Direct consequence of Lemma 439 (*other equivalent definition of set algebra*), and Lemma 413 (*closedness under set difference and local complement*). □

Lemma 441 *(intersection of set algebras).*
Let X and I be sets. Let $(\mathcal{A}_i)_{i\in I}$ be set algebras on X. Then, $\bigcap_{i\in I} \mathcal{A}_i$ is a set algebra on X.

Proof. Direct consequence of Definition 437 (*set algebra*), and **the definition of intersection**. □

Definition 442 *(generated set algebra).* Let X be a set. Let $G \subset \mathcal{P}(X)$. The *set algebra generated by* G is the intersection of all set algebras on X containing G; it is denoted $\mathcal{A}_X(G)$.

Lemma 443 *(generated set algebra is minimum).*
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{A}_X(G)$ is the smallest set algebra on X containing G.

Proof. Direct consequence of Definition 442 (*generated set algebra*), Lemma 441 (*intersection of set algebras*), and **properties of the intersection**. □

Lemma 444 ***(set algebra generation is monotone).***
Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_1 \subset G_2$. Then, we have $\mathcal{A}_X(G_1) \subset \mathcal{A}_X(G_2)$.

Proof. Lemma 443 (*generated set algebra is minimum*, with $G \stackrel{\text{def.}}{=} G_2$, then $G \stackrel{\text{def.}}{=} G_1$). □

Lemma 445 ***(set algebra generation is idempotent).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Then, $\mathcal{S}$ is a set algebra on X iff $\mathcal{A}_X(\mathcal{S}) = \mathcal{S}$.

Proof. Direct consequence of **reflexivity of inclusion**, Lemma 443 (*generated set algebra is minimum*, $\mathcal{S}$ and $\mathcal{A}_X(\mathcal{S})$ are both set algebras containing $\mathcal{S}$). □

Lemma 446 ***(partition of countable union in set algebra).***
Let X be a set. Let $\mathcal{A}$ be a set algebra on X. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Let $B_0 \stackrel{\text{def.}}{=} A_0$, and for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i\in[0..n]} B_i$. Then, we have

$$\forall n \in \mathbb{N}, \quad B_n \in \mathcal{A}, \tag{8.10}$$

$$\forall m, n \in \mathbb{N}, \quad m \neq n \implies B_m \cap B_n = \varnothing, \tag{8.11}$$

$$\forall n \in \mathbb{N}, \quad \bigcup_{i\in[0..n]} A_i = \biguplus_{i\in[0..n]} B_i \in \mathcal{A}. \tag{8.12}$$

Proof. For all $n \in \mathbb{N}$, let $P(n)$ be the property: $(\forall i \in \mathbb{N},\ i \leqslant n \Rightarrow B_i \in \mathcal{A}) \wedge \bigcup_{i\in[0..n]} B_i \in \mathcal{A}$.
Induction: $P(0)$. Trivial.
Induction: $P(n)$ implies $P(n+1)$. Direct consequence of Lemma 439 (*other equivalent definition of set algebra*, closedness under set difference), **associativity of intersection**, and Lemma 438 (*equivalent definition of set algebra*, closedness under union).
Hence, $P(n)$ holds for all $n \in \mathbb{N}$.

Therefore, from Lemma 215 (*partition of countable union*), all three properties hold. □

Lemma 447 ***(explicit set algebra).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G contains the full set, is closed under intersection, and satisfies the property:*

$$\forall A \in G,\ \exists B_1, B_2 \in G, \quad B_1 \cap B_2 = \varnothing \quad \wedge \quad A^c = B_1 \uplus B_2. \tag{8.13}$$

Then, the set algebra generated by G is the set of finite disjoint unions of elements of G:

$$\mathcal{A}_X(G) = \left\{ \biguplus_{p\in[0..n]} A_p \;\middle|\; n \in \mathbb{N} \quad \wedge \quad \forall p \in [0..n],\ A_p \in G \quad \wedge \quad \forall p, q \in [0..n],\ p \neq q \Rightarrow A_p \cap A_q = \varnothing \right\}. \tag{8.14}$$

Proof. Let $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(G)$. Let $\mathcal{U}$ be the set of finite disjoint unions of elements of G.

(0). $\mathcal{U} \subset \mathcal{A}$. From Lemma 443 (*generated set algebra is minimum*, $G \subset \mathcal{A}$), and Definition 437 (*set algebra*, $\mathcal{A}$ is closed under finite union), we have $\mathcal{U} \subset \mathcal{A}$.

(1a). $G \subset \mathcal{U}$. Direct consequence of the definition of $\mathcal{U}$ (with $n \stackrel{\text{def.}}{=} 0$).

(1b). $X \in \mathcal{U}$. Direct consequence of (1a).

(1c). $\mathcal{U}$ is closed under complement. Let $A \in \mathcal{U}$.

From the definition of $\mathcal{U}$, let $n \in \mathbb{N}$ and $(A_p)_{p\in[0..n]} \in G$ such that for all $p, q \in [0..n]$, $p \neq q$ implies $A_p \cap A_q = \varnothing$, and $A = \biguplus_{p\in[0..n]} A_p$. Let $p \in [0..n]$. Then, from (8.13), let $B_p^0, B_p^1 \in G$ such that $B_p^0 \cap B_p^1 = \varnothing$ and $A_p^c = B_p^0 \uplus B_p^1$.

Let $I \overset{\text{def.}}{=} \{0,1\}^{[0..n]}$ (its cardinality is 2^{n+1}). For all $\varphi \in I$, let $C^\varphi \overset{\text{def.}}{=} \bigcap_{p\in[0..n]} B_p^{\varphi(p)}$. Let $\varphi \in I$. Then, from Lemma 417 (*closedness under finite operations*, G is closed under finite intersection), we have $C^\varphi \in G$. Let $\psi \in I$. Assume that $\varphi \neq \psi$, i.e. there exists $q \in [0..n]$ such that $\varphi(q) \neq \psi(q)$. Then, from **the definition of intersection**, we have $C^\varphi \subset B_q^{\varphi(q)}$ and $C^\psi \subset B_q^{\psi(q)}$. Hence, from **monotonicity of intersection**, we have

$$C^\varphi \cap C^\psi \subset B_q^{\varphi(q)} \cap B_q^{\psi(q)} = \varnothing.$$

Moreover, from **De Morgan's laws**, and **distributivity of intersection over union**, we have

$$A^c = \bigcap_{p\in[0..n]} (B_p^0 \uplus B_p^1) = \bigcup_{\varphi\in I} \left(\bigcap_{p\in[0..n]} B_p^{\varphi(p)} \right) = \biguplus_{\varphi\in I} C^\varphi.$$

Thus, $A^c \in \mathcal{U}$. Hence, $\mathcal{U}$ is closed under complement.

(1d). $\boldsymbol{\mathcal{U}}$ is closed under intersection. Let $A^0, A^1 \in \mathcal{U}$.

Let $\alpha \in \{0,1\}$. From the definition of $\mathcal{U}$, let $n^\alpha \in \mathbb{N}$, let $(A^\alpha_{p^\alpha})_{p^\alpha\in[0..n^\alpha]} \in G$ such that for all $p^\alpha, q^\alpha \in [0..n^\alpha]$, $p^\alpha \neq q^\alpha$ implies $A^\alpha_{p^\alpha} \cap A^\alpha_{q^\alpha} = \varnothing$, and $A^\alpha = \biguplus_{p^\alpha\in[0..n^\alpha]} A^\alpha_{p^\alpha}$.

For all $p^0 \in [0..n^0]$, for all $p^1 \in [0..n^1]$, let $B_{p^0,p^1} \overset{\text{def.}}{=} A^0_{p^0} \cap A^1_{p^1} \in G$. Let $n \overset{\text{def.}}{=} n^0 n^1 + n^0 + n^1$. Let $\varphi : [0..n] \to [0..n^0] \times [0..n^1]$ be a bijection (their common cardinality is $n+1 = (n^0+1)(n^1+1)$). Let $p, q \in [0..n]$. Assume that $p \neq q$. Let $p^0, q^0 \in [0..n^0]$ and $p^1, q^1 \in [0..n^1]$ such that

$$(p^0, p^1) = \varphi(p) \quad \text{and} \quad (q^0, q^1) = \varphi(q).$$

Then, from **the definition of bijection and injection**, we have $p^0 \neq q^0$ or $p^1 \neq q^1$, i.e. $A^0_{p^0} \cap A^0_{q^0} = \varnothing$ or $A^1_{p^1} \cap A^1_{q^1} = \varnothing$. Thus, from **associativity and commutativity of intersection**, and since **$\varnothing$ is absorbing for intersection**, we have

$$B_{\varphi(p)} \cap B_{\varphi(q)} = (A^0_{p^0} \cap A^1_{p^1}) \cap (A^0_{q^0} \cap A^1_{q^1}) = (A^0_{p^0} \cap A^0_{q^0}) \cap (A^1_{p^1} \cap A^1_{q^1}) = \varnothing.$$

Moreover, from **left and right distributivity of intersection over union**, and **associativity and commutativity of union**, we have

$$A^0 \cap A^1 = \left(\bigcup_{p^0\in[0..n^0]} A^0_{p^0} \right) \cap \left(\bigcup_{p^1\in[0..n^1]} A^1_{p^1} \right) = \bigcup_{(p^0,p^1)\in[0..n^0]\times[0..n^1]} B_{p^0,p^1} = \biguplus_{p\in[0..n]} B_{\varphi(p)}.$$

Thus, $A^0 \cap A^1 \in \mathcal{U}$. Hence, $\mathcal{U}$ is closed under intersection.

(2). $\boldsymbol{\mathcal{A} \subset \mathcal{U}}$. From (1b), (1c), (1d), Lemma 438 (*equivalent definition of set algebra*, $\mathcal{U}$ is set algebra), (1a), and Lemma 443 (*generated set algebra is minimum*), we have $\mathcal{A} \subset \mathcal{U}$.

Therefore, from (0) and (2), we have $\mathcal{A} = \mathcal{U}$. □

8.4 Monotone class

Definition 448 ***(monotone class).*** Let X be a set. A subset $\mathcal{C}$ of $\mathcal{P}(X)$ is called *monotone class on* X iff it is closed under countable monotone union and intersection:

$$\forall (A_n)_{n\in\mathbb{N}} \in \mathcal{C}, \quad (\forall n \in \mathbb{N},\ A_n \subset A_{n+1}) \quad \Longrightarrow \quad \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C}, \tag{8.15}$$

$$\forall (A_n)_{n\in\mathbb{N}} \in \mathcal{C}, \quad (\forall n \in \mathbb{N},\ A_n \supset A_{n+1}) \quad \Longrightarrow \quad \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{C}. \tag{8.16}$$

Lemma 449 ***(intersection of monotone classes).*** *Let X and I be sets. Let $(\mathcal{C}_i)_{i\in I}$ be a family of monotone classes on X. Then, $\bigcap_{i\in I} \mathcal{C}_i$ is a monotone class on X.*

Proof.
Direct consequence of Definition 448 (*monotone class*), and **the definition of intersection**. □

Definition 450 ***(generated monotone class).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. The *monotone class generated by* G is the intersection of all monotone classes on X containing G; it is denoted by $\mathcal{C}_X(G)$.

Lemma 451 ***(generated monotone class is minimum).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{C}_X(G)$ is the smallest monotone class on X containing G.

Proof. Direct consequence of Definition 450 (*generated monotone class*), Lemma 449 (*intersection of monotone classes*), and **properties of the intersection**. □

Lemma 452 ***(monotone class generation is monotone).***
Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_1 \subset G_2$. Then, we have $\mathcal{C}_X(G_1) \subset \mathcal{C}_X(G_2)$.

Proof. Lemma 451 (*generated monotone class is minimum*, with $G \stackrel{\text{def.}}{=} G_2$, then $G \stackrel{\text{def.}}{=} G_1$). □

Lemma 453 ***(monotone class generation is idempotent).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Then, $\mathcal{S}$ is a monotone class on X iff $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$.

Proof. Direct consequence of **reflexivity of inclusion**, Lemma 451 (*generated monotone class is minimum*, $\mathcal{S}$ and $\mathcal{C}_X(\mathcal{S})$ are both monotone classes containing $\mathcal{S}$). □

Definition 454 ***(monotone class and symmetric set difference).*** Let X be a set. Let $\mathcal{C}$ be a monotone class on X. For all $A \subset X$, let $\mathcal{C}_A^{\setminus}$ be the subset system defined by

$$\mathcal{C}_A^{\setminus} \stackrel{\text{def.}}{=} \{B \subset X \mid A \setminus B \in \mathcal{C} \wedge B \setminus A \in \mathcal{C}\}. \tag{8.17}$$

Lemma 455 ***($\mathcal{C}^{\setminus}$ is symmetric).*** *Let X be a set. Let $\mathcal{C}$ be a monotone class on X. Then, for all $A, B \subset X$, we have $B \in \mathcal{C}_A^{\setminus}$ iff $A \in \mathcal{C}_B^{\setminus}$.*

Proof. Direct consequence of Definition 454 (*monotone class and symmetric set difference*). □

Lemma 456 ***($\mathcal{C}^{\setminus}$ is monotone class).*** *Let X be a set. Let $\mathcal{C}$ be a monotone class on X. Then, for all $A \subset X$, $\mathcal{C}_A^{\setminus}$ is a monotone class on X.*

Proof. Let $A \subset X$. Let $(B_n)_{n\in\mathbb{N}} \in \mathcal{C}_A^\setminus$. Let $n \in \mathbb{N}$. Then, from Definition 454 (*monotone class and symmetric set difference*), we have $A \setminus B_n \in \mathcal{C}$ and $B_n \setminus A \in \mathcal{C}$.

Assume first that, for all $n \in \mathbb{N}$, $B_n \subset B_{n+1}$. Let $n \in \mathbb{N}$. Then, from **monotonicity of set difference**, we have $A \setminus B_n \supset A \setminus B_{n+1}$ and $B_n \setminus A \subset B_{n+1} \setminus A$. Let $B \stackrel{\text{def.}}{=} \bigcup_{n\in\mathbb{N}} B_n$. Then, from **the definition of set difference**, **De Morgan's laws**, Definition 454 (*monotone class and symmetric set difference*), and Definition 448 (*monotone class*), we have $A \setminus B = \bigcap_{n\in\mathbb{N}}(A \setminus B_n) \in \mathcal{C}$ and $B \setminus A = \bigcup_{n\in\mathbb{N}}(B_n \setminus A) \in \mathcal{C}$. Thus, from Definition 454 (*monotone class and symmetric set difference*), we have $B \in \mathcal{C}_A^\setminus$. Hence, $\mathcal{C}_A^\setminus$ is closed under nondecreasing union.

Assume now that, for all $n \in \mathbb{N}$, $B_n \supset B_{n+1}$. Let $n \in \mathbb{N}$. Then, from **monotonicity of set difference**, we have $A \setminus B_n \subset A \setminus B_{n+1}$ and $B_n \setminus A \supset B_{n+1} \setminus A$. Let $B \stackrel{\text{def.}}{=} \bigcap_{n\in\mathbb{N}} B_n$. Then, from **the definition of set difference**, **De Morgan's laws**, Definition 454 (*monotone class and symmetric set difference*), and Definition 448 (*monotone class*), we have $A \setminus B = \bigcup_{n\in\mathbb{N}}(A \setminus B_n) \in \mathcal{C}$ and $B \setminus A = \bigcap_{n\in\mathbb{N}}(B_n \setminus A) \in \mathcal{C}$. Thus, from Definition 454 (*monotone class and symmetric set difference*), we have $B \in \mathcal{C}_A^\setminus$. Hence, $\mathcal{C}_A^\setminus$ is closed under nonincreasing intersection.

Therefore, from Definition 448 (*monotone class*), $\mathcal{C}_A^\setminus$ is a monotone class on X. □

Lemma 457 ***(monotone class is closed under set difference).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G is closed under set difference. Then, $\mathcal{C}_X(G)$ is closed under set difference.

Proof. For all $A \subset X$, we use the simplified notation $\mathcal{C}_A^\setminus \stackrel{\text{def.}}{=} (\mathcal{C}_X(G))_A^\setminus$.

(1). $\forall A \in G$, $\mathcal{C}_X(G) \subset \mathcal{C}_A^\setminus$. Let $A \in G$.
Let $B \in G$. Then, we have $A \setminus B,\ B \setminus A \in G$. Thus, from Lemma 451 (*generated monotone class is minimum*, $G \subset \mathcal{C}_X(G)$), and Definition 454 (*monotone class and symmetric set difference*), we have $B \in \mathcal{C}_A^\setminus$, i.e. $G \subset \mathcal{C}_A^\setminus$. Hence, from Lemma 456 (*$\mathcal{C}^\setminus$ is monotone class*), and Lemma 451 (*generated monotone class is minimum*), we have $\mathcal{C}_X(G) \subset \mathcal{C}_A^\setminus$.

(2). $\forall B \in \mathcal{C}_X(G)$, $\mathcal{C}_X(G) \subset \mathcal{C}_B^\setminus$. Let $B \in \mathcal{C}_X(G)$.
Let $A \in G$. Then, from (1), we have $B \in \mathcal{C}_A^\setminus$, and from Lemma 455 (*$\mathcal{C}^\setminus$ is symmetric*), we have $A \in \mathcal{C}_B^\setminus$, i.e. $G \subset \mathcal{C}_B^\setminus$. Hence, from Lemma 456 (*$\mathcal{C}^\setminus$ is monotone class*), and Lemma 451 (*generated monotone class is minimum*), we have $\mathcal{C}_X(G) \subset \mathcal{C}_B^\setminus$.

Let $A, B \in \mathcal{C}_X(G)$. Then, from (2), we have $B \in \mathcal{C}_A^\setminus$, and from Definition 454 (*monotone class and symmetric set difference*), we have $A \setminus B \in \mathcal{C}_X(G)$.

Therefore, $\mathcal{C}_X(G)$ is closed under set difference. □

Lemma 458 ***(monotone class generated by set algebra).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G is a set algebra on X. Then, $\mathcal{C}_X(G)$ is a set algebra on X.*

Proof. Direct consequence of Lemma 439 (*other equivalent definition of set algebra*, G contains the full set, and is closed under set difference), Lemma 451 (*generated monotone class is minimum*, $X \in G \subset \mathcal{C}_X(G)$), Lemma 457 (*monotone class is closed under set difference*, $\mathcal{C}_X(G)$ is closed under set difference), and Lemma 439 (*other equivalent definition of set algebra*). □

8.5 Lambda-system

Definition 459 ***(λ-system).***
Let X be a set. A subset Λ of $\mathcal{P}(X)$ is called *λ-system on X*, or *Dynkin-system on X* iff it contains the full set, it is closed under complement, and under countable disjoint union:

$$X \in \Lambda, \tag{8.18}$$

$$\forall A \in \Lambda, \quad A^c \in \Lambda, \tag{8.19}$$

$$\forall (A_n)_{n\in\mathbb{N}} \in \Lambda, \quad (\forall p, q \in \mathbb{N},\ p \neq q \Rightarrow A_p \cap A_q = \varnothing) \quad \Longrightarrow \quad \biguplus_{n\in\mathbb{N}} A_n \in \Lambda. \tag{8.20}$$

Lemma 460 ***(equivalent definition of λ-system).***
Let X be a set. Let $\Lambda \subset \mathcal{P}(X)$. Then, Λ is a λ-system on X iff (8.18) holds, and it is closed under local complement, and under countable monotone union:

$$\forall A, B \in \Lambda, \quad B \subset A \Rightarrow A \setminus B \in \Lambda, \tag{8.21}$$

$$\forall (A_n)_{n\in\mathbb{N}} \in \Lambda, \quad (\forall n \in \mathbb{N},\ A_n \subset A_{n+1}) \quad \Longrightarrow \quad \bigcup_{n\in\mathbb{N}} A_n \in \Lambda. \tag{8.22}$$

Proof. **"Left" implies "right".**
Direct consequence of Definition 459 (*λ-system*), Lemma 422 (*closedness under countable disjoint union and local complement*, Λ is closed under local complement), and Lemma 424 (*closedness under countable disjoint and monotone union*, Λ is closed under countable monotone union).

"Right" implies "left". Direct consequence of Lemma 411 (*closedness under local complement and complement*, Λ is closed under complement), and Lemma 425 (*closedness under countable monotone and disjoint union*, Λ is closed under countable disjoint union). □

Lemma 461 ***(other properties of λ-system).***
Let X be a set. Let Λ be a λ-system on X. Then, Λ is nonempty, contains the empty set, and is closed under countable monotone intersection.

Proof. Direct consequence of Definition 459 (*λ-system*, closedness under complement), Lemma 408 (*nonempty and with empty or full*), Lemma 409 (*with empty and full*), Lemma 460 (*equivalent definition of λ-system*), and Lemma 423 (*closedness under countable union and intersection*). □

Lemma 462 ***(intersection of λ-systems).***
Let X and I be sets. Let $(\Lambda_i)_{i\in I}$ be λ-systems on X. Then, if $\bigcap_{i\in I} \Lambda_i$ is a λ-system on X.

Proof. Direct consequence of Definition 459 (*λ-system*), and **the definition of intersection**. □

Definition 463 ***(generated λ-system).*** Let X be a set. Let $G \subset \mathcal{P}(X)$. The *λ-system generated by G* is the intersection of all λ-systems on X containing G; it is denoted $\Lambda_X(G)$.

Lemma 464 ***(generated λ-system is minimum).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, $\Lambda_X(G)$ is the smallest λ-system on X containing G.

Proof. Direct consequence of Definition 463 (*generated λ-system*), Lemma 462 (*intersection of λ-systems*), and **properties of the intersection**. □

Lemma 465 ***(λ-system generation is monotone).***
Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_1 \subset G_2$. Then, we have $\Lambda_X(G_1) \subset \Lambda_X(G_2)$.

Proof. Lemma 464 (*generated λ-system is minimum*, with $G \stackrel{\text{def.}}{=} G_2$, then $G \stackrel{\text{def.}}{=} G_1$). □

Lemma 466 ***(λ-system generation is idempotent).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Then, $\mathcal{S}$ is a λ-system on X iff $\Lambda_X(\mathcal{S}) = \mathcal{S}$.

Proof. Direct consequence of **reflexivity of inclusion**, Lemma 464 (*generated λ-system is minimum*, $\mathcal{S}$ and $\Lambda_X(\mathcal{S})$ are both λ-systems containing $\mathcal{S}$). □

Definition 467 ***(λ-system and intersection).*** Let X be a set.
Let Λ be a λ-system on X. For all $A \subset X$, let $\Lambda_A^\cap$ be the subset system defined by

$$\Lambda_A^\cap \stackrel{\text{def.}}{=} \{B \in \Lambda \mid A \cap B \in \Lambda\}. \tag{8.23}$$

Lemma 468 ***($\Lambda^\cap$ is symmetric).***
Let X be a set. Let Λ be a λ-system on X. Then, for all $A, B \in \Lambda$, we have $B \in \Lambda_A^\cap$ iff $A \in \Lambda_B^\cap$.

Proof. Direct consequence of Definition 467 (*λ-system and intersection*). □

Lemma 469 ***($\Lambda^\cap$ is λ-system).***
Let X be a set. Let Λ be a λ-system on X. Then, for all $A \in \Lambda$, $\Lambda_A^\cap$ is a λ-system on X.

Proof. Let $A \in \Lambda$.

From Definition 459 (*λ-system*, $X \in \Lambda$), and **the identity $A \cap X = A \in \Lambda$**, we have $X \in \Lambda_A^\cap$.

Let $B, C \in \Lambda_A^\cap$, i.e. $B, C, A \cap B, A \cap C \in \Lambda$. Assume that $C \subset B$.
Then, from Lemma 460 (*equivalent definition of λ-system*, closedness under local complement), we have $B \setminus C \in \Lambda$. Moreover, from **distributivity of intersection over set difference**, **monotonicity of intersection**, and Lemma 460 (*equivalent definition of λ-system*, closedness under local complement), we have $(A \cap C) \subset (A \cap B)$ and $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) \in \Lambda$. Hence, $\Lambda_A^\cap$ is closed under local complement.

Let $(B_n)_{n \in \mathbb{N}} \in \Lambda_A^\cap$, i.e. for all $n \in \mathbb{N}$, $B_n, A \cap B_n \in \Lambda$. Assume that for all $n \in \mathbb{N}$, $B_n \subset B_{n+1}$. Then, from Lemma 460 (*equivalent definition of λ-system*, closedness under countable monotone union), we have $\bigcup_{n \in \mathbb{N}} B_n \in \Lambda$. Moreover, from **distributivity of intersection over union**, **monotonicity of intersection**, and Lemma 460 (*equivalent definition of λ-system*, closedness under countable monotone union), we have

$$(A \cap B_n) \subset (A \cap B_{n+1}) \quad \text{and} \quad A \cap \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \Lambda.$$

Hence, $\Lambda_A^\cap$ is closed under countable monotone union.

Therefore, from Lemma 460 (*equivalent definition of λ-system*), $\Lambda_A^\cap$ is a λ-system on X. □

Lemma 470 ***(λ-system with intersection).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G is closed under intersection. Then, $\forall A \in \Lambda_X(G)$, $(\Lambda_X(G))_A^\cap = \Lambda_X(G)$.*

Proof. For all $A \subset X$, we use the simplified notation $\Lambda_A^\cap \stackrel{\text{def.}}{=} (\Lambda_X(G))_A^\cap$.

(0). From Definition 467 (*λ-system and intersection*), we have $\forall A \in \Lambda_X(G)$, $\Lambda_A^\cap \subset \Lambda_X(G)$.

(1). $\forall A \in G$, $\Lambda_X(G) \subset \Lambda_A^\cap$. Let $A \in G$.
Let $B \in G$. Then, from Lemma 464 (*generated λ-system is minimum*, $G \subset \Lambda_X(G)$), and Definition 467 (*λ-system and intersection*), we have $B \in \Lambda_A^\cap$, i.e. $G \subset \Lambda_A^\cap$. Hence, from Lemma 469 (*$\Lambda^\cap$ is λ-system*, with $A \in G \subset \Lambda_X(G)$), and Lemma 464 (*generated λ-system is minimum*), we have $\Lambda_X(G) \subset \Lambda_A^\cap$.

(2). $\forall B \in \Lambda_X(G)$, $\Lambda_X(G) \subset \Lambda_B^\cap$. Let $B \in \Lambda_X(G)$.
Let $A \in G$. Then, from (1), we have $B \in \Lambda_X(G) \subset \Lambda_A^\cap$. Thus, from Lemma 464 (*generated λ-system is minimum*, $G \subset \Lambda_X(G)$), and Lemma 468 (*$\Lambda^\cap$ is symmetric*, with $A, B \in \Lambda_X(G)$), we have $A \in \Lambda_B^\cap$, i.e. $G \subset \Lambda_B^\cap$. Hence, from Lemma 469 (*$\Lambda^\cap$ is λ-system*), and Lemma 464 (*generated λ-system is minimum*), we have $\Lambda_X(G) \subset \Lambda_B^\cap$.

Therefore, from (0) and (2), we have the equality. □

Lemma 471 (*λ-system is closed under intersection*). *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G is closed under intersection. Then, $\Lambda_X(G)$ is closed under intersection.*

Proof. Let $A, B \in \Lambda_X(G)$. Then, from Lemma 470 (*λ-system with intersection*, $\underline{B \text{ belongs to } \Lambda_X(G) = (\Lambda_X(G))^{\cap}_A}$), and Definition 467 (*$\lambda$-system and intersection*), we have $A \cap B \in \Lambda_X(G)$.

Therefore, $\Lambda_X(G)$ is closed under intersection. □

Lemma 472 (*λ-system generated by π-system*). *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that G is a π-system on X. Then, $\Lambda_X(G)$ is a π-system on X.*

Proof. Direct consequence of Definition 428 (*π-system*, $\underline{G \neq \varnothing \text{ and } G \text{ is closed under finite intersection}}$), Lemma 417 (*closedness under finite operations*, $\underline{G \text{ is closed under intersection}}$), Lemma 464 (*generated λ-system is minimum*, $\underline{\varnothing \neq G \subset \Lambda_X(G)}$), Lemma 471 (*$\lambda$-system is closed under intersection*, $\underline{\Lambda_X(G) \text{ is closed under intersection}}$), Lemma 417 (*closedness under finite operations*, $\underline{\Lambda_X(G) \text{ is closed under finite intersection}}$), and Definition 428 (*$\pi$-system*). □

8.6 Sigma-algebra

Remark 473. Note that the following concept of “σ-algebra” is sometimes called “σ-field”.

Definition 474 *(σ-algebra).* Let X be a set. A subset Σ of $\mathcal{P}(X)$ is called *σ-algebra on X* iff it contains the empty set, it is closed under complement, and under countable union:

$$\varnothing \in \Sigma, \tag{8.24}$$

$$\forall A \in \Sigma, \quad A^c \in \Sigma, \tag{8.25}$$

$$\forall I \subset \mathbb{N},\ \forall (A_i)_{i\in I} \in \Sigma, \quad \bigcup_{i\in I} A_i \in \Sigma. \tag{8.26}$$

Lemma 475 *(equivalent definition of σ-algebra).*
Let X be a set. Let $\Sigma \subset \mathcal{P}(X)$. Then, Σ is a σ-algebra on X iff (8.25) holds and

$$\varnothing \in \Sigma \quad \vee \quad X \in \Sigma \quad \vee \quad \Sigma \neq \varnothing, \tag{8.27}$$

$$\forall I \subset \mathbb{N},\ \forall (A_i)_{i\in I} \in \Sigma,\ \bigcup_{i\in I} A_i \in \Sigma \quad \vee \quad \forall I \subset \mathbb{N},\ \forall (A_i)_{i\in I} \in \Sigma,\ \bigcap_{i\in I} A_i \in \Sigma. \tag{8.28}$$

Proof. Direct consequence of Definition 474 (*σ-algebra*), Lemma 408 (*nonempty and with empty or full*), Lemma 409 (*with empty and full*), and Lemma 423 (*closedness under countable union and intersection*). □

Remark 476. Note that from the previous lemma, we may define σ-algebras as subset systems that satisfies (8.25), and any term in each of the disjunctions (8.27) and (8.28).

Lemma 477 *(σ-algebra is set algebra).*
Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is a set algebra on X.

Proof. Direct consequence of Definition 437 (*set algebra*), and Definition 474 (*σ-algebra*). □

Lemma 478 *(σ-algebra is closed under set difference).* *Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is closed under set difference and local complement.*

Proof. Direct consequence of Lemma 477 (*σ-algebra is set algebra*), Lemma 439 (*other equivalent definition of set algebra*), and Lemma 440 (*set algebra is closed under local complement*). □

Lemma 479 *(other properties of σ-algebra).* *Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is closed under countable monotone intersection and union, and countable disjoint union.*

Proof. Direct consequence of Lemma 475 (*equivalent definition of σ-algebra*). □

Lemma 480 *(partition of countable union in σ-algebra).*
Let X be a set. Let Σ be a σ-algebra on X. Let $(A_n)_{n\in\mathbb{N}} \subset \Sigma$. Let $B_0 \stackrel{\text{def.}}{=} A_0$, and for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i\in[0..n]} B_i$. Then, we have

$$\forall n \in \mathbb{N}, \quad B_n \in \Sigma, \tag{8.29}$$

$$\forall m, n \in \mathbb{N}, \quad m \neq n \quad \Longrightarrow \quad B_m \cap B_n = \varnothing, \tag{8.30}$$

$$\forall n \in \mathbb{N}, \quad \bigcup_{i\in[0..n]} A_i = \biguplus_{i\in[0..n]} B_i \in \Sigma, \tag{8.31}$$

$$\bigcup_{i\in\mathbb{N}} A_i = \biguplus_{i\in\mathbb{N}} B_i \in \Sigma. \tag{8.32}$$

Proof. Direct consequence of Lemma 477 (*σ-algebra is set algebra*), Lemma 446 (*partition of countable union in set algebra*), Lemma 215 (*partition of countable union*), and Definition 474 (*σ-algebra*, closedness under countable union with $I = \mathbb{N}$). ☐

Lemma 481 ***(intersection of σ-algebras).***
Let X and I be sets. Let $(\Sigma_i)_{i\in I}$ be σ-algebras on X. Then, $\bigcap_{i\in I} \Sigma_i$ is a σ-algebra on X.

Proof. Direct consequence of Definition 474 (*σ-algebra*), and **the definition of intersection**. ☐

Definition 482 ***(generated σ-algebra).*** Let X be a set. Let $G \subset \mathcal{P}(X)$. The *σ-algebra generated by G* is the intersection of all σ-algebras on X containing G; it is denoted $\Sigma_X(G)$.

Lemma 483 ***(generated σ-algebra is minimum).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, $\Sigma_X(G)$ is the smallest σ-algebra on X containing G.

Proof. Direct consequence of Definition 482 (*generated σ-algebra*), Lemma 481 (*intersection of σ-algebras*), and **properties of the intersection**. ☐

Lemma 484 ***(σ-algebra generation is monotone).***
Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_1 \subset G_2$. Then, we have $\Sigma_X(G_1) \subset \Sigma_X(G_2)$.

Proof. Lemma 483 (*generated σ-algebra is minimum*, with $G \stackrel{\text{def.}}{=} G_2$, then $G \stackrel{\text{def.}}{=} G_1$). ☐

Lemma 485 ***(σ-algebra generation is idempotent).***
Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Then, $\mathcal{S}$ is a σ-algebra on X iff $\Sigma_X(\mathcal{S}) = \mathcal{S}$.

Proof. Direct consequence of **reflexivity of inclusion**, Lemma 483 (*generated σ-algebra is minimum*, $\mathcal{S}$ and $\Sigma_X(\mathcal{S})$ are both σ-algebras containing $\mathcal{S}$). ☐

Lemma 486 ***(σ-algebra is π-system).***
Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is a π-system on X.

Proof. Direct consequence of Definition 428 (*π-system*), and Lemma 475 (*equivalent definition of σ-algebra*). ☐

Lemma 487 ***(σ-algebra contains π-system).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Pi_X(G) \subset \Sigma_X(G)$.

Proof. Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $G \subset \Sigma_X(G)$), Lemma 486 (*σ-algebra is π-system*), and Lemma 433 (*generated π-system is minimum*). ☐

Lemma 488 ***(π-system contains σ-algebra).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$, and that $\Pi_X(G)$ is closed under complement and countable disjoint union. Then, we have $\Sigma_X(G) \subset \Pi_X(G)$.

Proof. Direct consequence of Lemma 433 (*generated π-system is minimum*, $\Pi_X(G)$ is a π-system, and $G \subset \Pi_X(G)$), Definition 428 (*π-system*, $\Pi_X(G)$ is nonempty, and closed under intersection), Lemma 408 (*nonempty and with empty or full*, $\varnothing \in \Pi_X(G)$), Lemma 426 (*closedness under countable disjoint union and countable union*, $\Pi_X(G)$ is closed under countable union), Definition 474 (*σ-algebra*, $\Pi_X(G)$ is a σ-algebra), and Lemma 483 (*generated σ-algebra is minimum*). ☐

Lemma 489 ***(σ-algebra generated by π-system).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Sigma_X(\Pi_X(G)) = \Sigma_X(G)$.

Proof. Direct consequence of Lemma 487 (*σ-algebra contains π-system*, $\Pi_X(G) \subset \Sigma_X(G)$), Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $\Sigma_X(\Pi_X(G)) \subset \Sigma_X(G)$), Lemma 433 (*generated π-system is minimum*, $G \subset \Pi_X(G)$), and Lemma 484 (*σ-algebra generation is monotone*, $\Sigma_X(G)$ is included in $\Sigma_X(\Pi_X(G))$). □

Lemma 490 ***(σ-algebra contains set algebra).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{A}_X(G) \subset \Sigma_X(G)$.

Proof. Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $G \subset \Sigma_X(G)$), Lemma 477 (*σ-algebra is set algebra*), and Lemma 443 (*generated set algebra is minimum*). □

Lemma 491 ***(set algebra contains σ-algebra).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{A}_X(G)$ is closed under countable monotone union. Then, we have $\Sigma_X(G) \subset \mathcal{A}_X(G)$.*

Proof. Direct consequence of Lemma 443 (*generated set algebra is minimum*, $\mathcal{A}_X(G)$ is a set algebra, and $G \subset \mathcal{A}_X(G)$), Definition 437 (*set algebra*, $\mathcal{A}_X(G)$ contains $\varnothing$, is closed under complement and union), Lemma 427 (*closedness under countable monotone union and countable union*, $\mathcal{A}_X(G)$ is closed under countable union), Definition 474 (*σ-algebra*, $\mathcal{A}_X(G)$ is a σ-algebra), and Lemma 483 (*generated σ-algebra is minimum*). □

Lemma 492 ***(σ-algebra generated by set algebra).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_X(\mathcal{A}_X(G)) = \Sigma_X(G)$.

Proof. Direct consequence of Lemma 490 (*σ-algebra contains set algebra*, $\mathcal{A}_X(G) \subset \Sigma_X(G)$), Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $\Sigma_X(\mathcal{A}_X(G)) \subset \Sigma_X(G)$), Lemma 443 (*generated set algebra is minimum*, $G \subset \mathcal{A}_X(G)$), and Lemma 484 (*$\sigma$-algebra generation is monotone*, $\Sigma_X(G)$ is included in $\Sigma_X(\mathcal{A}_X(G))$). □

Lemma 493 ***(σ-algebra is monotone class).***
Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is a monotone class on X.

Proof. Direct consequence of Definition 448 (*monotone class*), and Lemma 479 (*other properties of σ-algebra*). □

Lemma 494 ***(σ-algebra contains monotone class).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{C}_X(G) \subset \Sigma_X(G)$.

Proof. Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $G \subset \Sigma_X(G)$), Lemma 493 (*σ-algebra is monotone class*), and Lemma 451 (*generated monotone class is minimum*). □

Lemma 495 ***(monotone class contains σ-algebra).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{C}_X(G)$ contains the empty set, and is closed under complement and union. Then, we have $\Sigma_X(G) \subset \mathcal{C}_X(G)$.

Proof. Direct consequence of Lemma 451 (*generated monotone class is minimum*, $\mathcal{C}_X(G)$ is a monotone class, and $G \subset \mathcal{C}_X(G)$), Definition 448 (*monotone class*, $\mathcal{C}_X(G)$ is closed under countable monotone union), Lemma 427 (*closedness under countable monotone union and countable union*, $\mathcal{C}_X(G)$ is closed under countable union), Definition 474 (*σ-algebra*, $\mathcal{C}_X(G)$ is a σ-algebra), and Lemma 483 (*generated σ-algebra is minimum*). □

Lemma 496 ***(σ-algebra generated by monotone class).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_X(\mathcal{C}_X(G)) = \Sigma_X(G)$.

Proof. Direct consequence of Lemma 494 (*σ-algebra contains monotone class*, $\mathcal{C}_X(G) \subset \Sigma_X(G)$), Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $\Sigma_X(\mathcal{C}_X(G)) \subset \Sigma_X(G)$), Lemma 451 (*generated monotone class is minimum*, $G \subset \mathcal{C}_X(G)$), and Lemma 484 (*$\sigma$-algebra generation is monotone*, $\Sigma_X(G) \subset \Sigma_X(\mathcal{C}_X(G))$). □

Lemma 497 ***(σ-algebra is λ-system).***
Let X be a set. Let Σ be a σ-algebra on X. Then, Σ is a λ-system on X.

Proof. Direct consequence of Definition 459 (*λ-system*), Lemma 475 (*equivalent definition of σ-algebra*), and Lemma 479 (*other properties of σ-algebra*). □

Lemma 498 ***(σ-algebra contains λ-system).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Lambda_X(G) \subset \Sigma_X(G)$.

Proof. Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $G \subset \Sigma_X(G)$), Lemma 497 (*σ-algebra is λ-system*), and Lemma 464 (*generated λ-system is minimum*). □

Lemma 499 ***(λ-system contains σ-algebra).*** *Let X be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\Lambda_X(G)$ is closed under intersection. Then, we have $\Sigma_X(G) \subset \Lambda_X(G)$.*

Proof. Direct consequence of Lemma 464 (*generated λ-system is minimum*, $\Lambda_X(G)$ is a λ-system, and $G \subset \Lambda_X(G)$), Definition 459 (*λ-system*, $\Lambda_X(G)$ contains the full set, is closed under complement and countable disjoint union), Lemma 426 (*closedness under countable disjoint union and countable union*, $\Lambda_X(G)$ is closed under countable union), Lemma 475 (*equivalent definition of σ-algebra*, $\Lambda_X(G)$ is a σ-algebra), and Lemma 483 (*generated σ-algebra is minimum*). □

Lemma 500 ***(σ-algebra generated by λ-system).***
Let X be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_X(\Lambda_X(G)) = \Sigma_X(G)$.

Proof. Direct consequence of Lemma 498 (*σ-algebra contains λ-system*, $\Lambda_X(G) \subset \Sigma_X(G)$), Lemma 483 (*generated σ-algebra is minimum*, $\Sigma_X(G)$ is a σ-algebra, and $\Sigma_X(\Lambda_X(G)) \subset \Sigma_X(G)$), Lemma 464 (*generated λ-system is minimum*, $G \subset \Lambda_X(G)$), and Lemma 484 (*σ-algebra generation is monotone*, $\Sigma_X(G)$ is included in $\Sigma_X(\Lambda_X(G))$). □

Lemma 501 ***(other σ-algebra generator).*** *Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_1 \subset \Sigma_X(G_2)$ and $G_2 \subset \Sigma_X(G_1)$. Then, we have $\Sigma_X(G_1) = \Sigma_X(G_2)$.*

Proof. From Lemma 484 (*σ-algebra generation is monotone*), and Lemma 485 (*σ-algebra generation is idempotent*), we have

$$\Sigma_X(G_1) \subset \Sigma_X(\Sigma_X(G_2)) = \Sigma_X(G_2) \quad \text{and} \quad \Sigma_X(G_2) \subset \Sigma_X(\Sigma_X(G_1)) = \Sigma_X(G_1).$$

Therefore, we have $\Sigma_X(G_1) = \Sigma_X(G_2)$. □

Lemma 502 ***(complete generated σ-algebra).*** *Let X be a set. Let $G_1, G_2 \subset \mathcal{P}(X)$. Assume that $G_2 \subset \Sigma_X(G_1)$. Then, we have $\Sigma_X(G_1 \cup G_2) = \Sigma_X(G_1)$.*

Proof. From Lemma 483 (*generated σ-algebra is minimum*), and Lemma 484 (*σ-algebra generation is monotone*), we have $G_1 \subset \Sigma_X(G_1) \subset \Sigma_X(G_1 \cup G_2)$. From Lemma 483 (*generated σ-algebra is minimum*, $G_1 \subset \Sigma_X(G_1)$), and **monotonicity of union**, we have $G_1 \cup G_2 \subset \Sigma_X(G_1)$.

Therefore, from Lemma 501 (*other σ-algebra generator*, with G_1 and $G_1 \cup G_2$), we have $\Sigma_X(G_1 \cup G_2) = \Sigma_X(G_1)$. □

Lemma 503 ***(countable σ-algebra generator).***
Let X be a set. Let $G_1 \subset G_2 \subset \mathcal{P}(X)$. Assume that all elements of G_2 are countable unions of elements of G_1. Then, we have $\Sigma_X(G_1) = \Sigma_X(G_2)$.

Proof. Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $G_1 \subset G_2 \subset \Sigma_X(G_2)$), Definition 474 (*$\sigma$-algebra*, closedness under countable union, thus $G_2 \subset \Sigma_X(G_1)$), and Lemma 501 (*other σ-algebra generator*). □

Remark 504. Note that in the following proof, the point (1) does not depend on the hypotheses. It could be an independent lemma that characterizes $\Sigma_1 \overline{\times} \Sigma_2$.

Note also that $\Sigma_1 \overline{\times} \Sigma_2$ may not be a σ-algebra. See Section 9.4 for a definition of a σ-algebra on the product of measurable spaces.

Lemma 505 *(set algebra generated by product of σ-algebras).* *Let X_1 and X_2 be sets. For all $i \in \{1,2\}$, let Σ_i be a σ-algebra on X_i. Let $X \overset{\text{def.}}{=} X_1 \times X_2$, and $\overline{\Sigma} \overset{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$. Then, $\mathcal{A}_X(\overline{\Sigma})$ is the set of finite disjoint unions of elements of $\overline{\Sigma}$.*

Proof. **(1). $X \in \overline{\Sigma}$.** Direct consequence of Definition 217 (*product of subsets of parties*), and Lemma 475 (*equivalent definition of σ-algebra*, Σ_1 and Σ_2 contain full set.).

(2). $\overline{\Sigma}$ is closed under intersection. Let $A, B \in \overline{\Sigma}$.
From Definition 217 (*product of subsets of parties*), let $A_1, B_1 \in \Sigma_1$ and $A_2, B_2 \in \Sigma_2$ such that $A = A_1 \times A_2$ and $B = B_1 \times B_2$. Then, from **compatibility of intersection with Cartesian product**, and Lemma 475 (*equivalent definition of σ-algebra*, closedness under intersection for Σ_1 and Σ_2), we have $A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2)$ with $A_1 \cap B_1 \in \Sigma_1$ and $A_2 \cap B_2 \in \Sigma_2$. Thus, $A \cap B \in \overline{\Sigma}$. Hence, $\overline{\Sigma}$ is closed under intersection.

(3). $\forall A \in \overline{\Sigma}, \exists B, C \in \overline{\Sigma}, B \cap C = \varnothing \wedge A^c = B \uplus C$. Let $A \in \overline{\Sigma}$.
From Definition 217 (*product of subsets of parties*), let $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$ such that $A = A_1 \times A_2$. Let $B \overset{\text{def.}}{=} X_1 \times A_2^c$ and $C \overset{\text{def.}}{=} A_1^c \times A_2$. Then, from Definition 217 (*product of subsets of parties*), Lemma 475 (*equivalent definition of σ-algebra*, Σ_1 contains full set, and closedness under complement for Σ_1 and Σ_2), and **set operations properties**, we have $B, C \in \overline{\Sigma}$, $B \cap C = \varnothing$, and $A^c = B \uplus C$.

Therefore, from (1), (2), (3), and Lemma 447 (*explicit set algebra*, with $G \overset{\text{def.}}{=} \overline{\Sigma}$), $\mathcal{A}_X(\overline{\Sigma})$ is the set of finite disjoint unions of elements of $\overline{\Sigma}$. □

8.7 Dynkin π–λ theorem

Lemma 506 ***(π-system and λ-system is σ-algebra).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is a π-system on X and a λ-system on X. Then, $\mathcal{S}$ is a σ-algebra on X.*

Proof. Direct consequence of Lemma 466 (*λ-system generation is idempotent*, $\underline{\Lambda_X(\mathcal{S}) = \mathcal{S}}$), Definition 428 (*$\pi$-system*, $\underline{\Lambda_X(\mathcal{S})}$ is closed under intersection), Lemma 499 (*λ-system contains σ-algebra*, $\underline{\Sigma_X(\mathcal{S})}$ is included in $\Lambda_X(\mathcal{S})$), Lemma 498 (*σ-algebra contains λ-system*, $\underline{\Lambda_X(\mathcal{S}) \subset \Sigma_X(\mathcal{S})}$), and Lemma 485 (*$\sigma$-algebra generation is idempotent*, $\underline{\Sigma_X(\mathcal{S}) = \Lambda_X(\mathcal{S}) = \mathcal{S}}$, thus $\mathcal{S}$ is a σ-algebra). $\square$

Remark 507. See the sketch of next proof in Section 5.7.

Theorem 508 ***(Dynkin π–λ theorem).***
Let X be a set. Let Π be a π-system on X. Then, we have $\Lambda_X(\Pi) = \Sigma_X(\Pi)$.

Proof. Direct consequence of Lemma 472 (*λ-system generated by π-system*, $\underline{\Lambda_X(\Pi)}$ is π-system), Lemma 464 (*generated λ-system is minimum*, $\underline{\Lambda_X(\Pi)}$ is λ-system), Lemma 506 (*π-system and λ-system is σ-algebra*, $\underline{\Lambda_X(\Pi)}$ is σ-algebra), Lemma 485 (*σ-algebra generation is idempotent*, $\underline{\Sigma_X(\Lambda_X(\Pi)) = \Lambda_X(\Pi)}$), and Lemma 500 (*$\sigma$-algebra generated by λ-system*, $\underline{\Sigma_X(\Lambda_X(\Pi)) = \Sigma_X(\Pi)}$). $\square$

Remark 509. Note that the Dynkin π–λ theorem may take the following form: if a λ-system contains a π-system, then it also contains the σ-algebra generated by the π-system.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemma 668 in Section 11.3, itself later used to prove uniqueness of the Lebesgue measure.

Lemma 510 ***(usage of Dynkin π–λ theorem).*** *Let X be a set. Let Σ be a σ-algebra on X. Let P be a predicate over $\mathcal{P}(X)$, and let $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \,|\, P(A)\}$. Let $G \subset \mathcal{P}(X)$ be a nonempty generator of Σ. Assume that $\Pi_X(G) \subset \mathcal{S}$, and that $\mathcal{S}$ is a λ-system.*
Then, we have $\mathcal{S} = \Sigma$, i.e. P holds for all subsets in Σ.

Proof. From the definition of $\mathcal{S}$, we have $\mathcal{S} \subset \Sigma$.

Let $\Pi \stackrel{\text{def.}}{=} \Pi_X(G)$. Then, from Definition 482 (*generated σ-algebra*, $\underline{\Sigma = \Sigma_X(G)}$), Lemma 433 (*generated π-system is minimum*, $\underline{G \subset \Pi}$, and Π is a π-system), Lemma 484 (*σ-algebra generation is monotone*, $\underline{\Sigma_X(G)}$ is included in $\Sigma_X(\Pi)$), Theorem 508 (*Dynkin π–λ theorem*, $\underline{\Sigma_X(\Pi) = \Lambda_X(\Pi)}$), Lemma 465 (*$\lambda$-system generation is monotone*, $\underline{\Lambda_X(\Pi) \subset \Lambda_X(\mathcal{S})}$), and Lemma 466 (*$\lambda$-system generation is idempotent*, $\underline{\Lambda_X(\mathcal{S}) = \mathcal{S}}$), we have $\Sigma \subset \mathcal{S}$.

Therefore, we have the equality, i.e. P holds for all subsets in Σ. $\square$

8.8 Monotone class theorem

Lemma 511 ***(algebra and monotone class is σ-algebra).*** *Let X be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is a set algebra on X and a monotone class on X. Then, $\mathcal{S}$ is a σ-algebra on X.*

Proof. Direct consequence of Lemma 453 (*monotone class generation is idempotent*, $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$), Definition 437 (*set algebra*, $\mathcal{C}_X(\mathcal{S})$ contains the empty set, and is closed under complement and union), Lemma 495 (*monotone class contains σ-algebra*, $\Sigma_X(\mathcal{S}) \subset \mathcal{C}_X(\mathcal{S})$), Lemma 494 (*$\sigma$-algebra contains monotone class*, $\mathcal{C}_X(\mathcal{S}) \subset \Sigma_X(\mathcal{S})$), and Lemma 485 (*$\sigma$-algebra generation is idempotent*, $\Sigma_X(\mathcal{S}) = \mathcal{C}_X(\mathcal{S}) = \mathcal{S}$, thus $\mathcal{S}$ is σ-algebra.). □

Remark 512. See the sketch of next proof in Section 5.7.

Theorem 513 ***(monotone class).***
Let X be a set. Let $\mathcal{A}$ be a set algebra on X. Then, $\mathcal{C}_X(\mathcal{A}) = \Sigma_X(\mathcal{A})$.

Proof. Direct consequence of Lemma 458 (*monotone class generated by set algebra*, $\mathcal{C}_X(\mathcal{A})$ is set algebra), Lemma 451 (*generated monotone class is minimum*, $\mathcal{C}_X(\mathcal{A})$ is monotone class), Lemma 511 (*algebra and monotone class is σ-algebra*, $\mathcal{C}_X(\mathcal{A})$ is σ-algebra), Lemma 485 (*σ-algebra generation is idempotent*, thus $\Sigma_X(\mathcal{C}_X(\mathcal{A})) = \mathcal{C}_X(\mathcal{A})$), and Lemma 496 (*$\sigma$-algebra generated by monotone class*, $\Sigma_X(\mathcal{C}_X(\mathcal{A})) = \Sigma_X(\mathcal{A})$). □

Remark 514. Note that the monotone class theorem may take the following form: if a monotone class contains a set algebra, then it also contains the σ-algebra generated by the algebra.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemmas 827 and 837 in Section 13.4 in the context of product spaces.

Lemma 515 ***(usage of monotone class theorem).*** *Let X be a set. Let Σ be a σ-algebra on X. Let P be a predicate over $\mathcal{P}(X)$, and let $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \,|\, P(A)\}$. Let $G \subset \mathcal{P}(X)$ be a generator of Σ. Assume that $\mathcal{A}_X(G) \subset \mathcal{S}$, and that $\mathcal{S}$ is a monotone class.*
Then, we have $\mathcal{S} = \Sigma$, i.e. P holds for all subsets in Σ.

Proof. From the definition of $\mathcal{S}$, we have $\mathcal{S} \subset \Sigma$.

Let $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(G)$. Then, from Definition 482 (*generated σ-algebra*, $\Sigma = \Sigma_X(G)$), Lemma 443 (*generated set algebra is minimum*, $G \subset \mathcal{A}$, and $\mathcal{A}$ is set algebra), Lemma 484 (*σ-algebra generation is monotone*, $\Sigma_X(G) \subset \Sigma_X(\mathcal{A})$), Theorem 513 (*monotone class*, $\Sigma_X(\mathcal{A}) = \mathcal{C}_X(\mathcal{A})$), Lemma 452 (*monotone class generation is monotone*, $\mathcal{C}_X(\mathcal{A}) \subset \mathcal{C}_X(\mathcal{S})$), and Lemma 453 (*monotone class generation is idempotent*, $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$), we have $\Sigma \subset \mathcal{S}$.

Therefore, we have the equality, i.e. P holds for all subsets in Σ. □

Chapter 9

Measurability

Contents

9.1 Measurable space and Borel subsets

Definition 516 *(measurable space).* Let X be a set. Let Σ be a σ-algebra on X. Then, (X, Σ) is called *measurable space*, and elements of Σ are said *(Σ-)measurable.*

Definition 517 *(Borel σ-algebra).*
Let $(X, \mathcal{T})$ be a topological space. The *Borel σ-algebra of X* is the σ-algebra generated by the open subsets; it is denoted $\mathcal{B}(X) \stackrel{\text{def.}}{=} \Sigma_X(\mathcal{T})$.

$\mathcal{B}(X)$-measurable subsets are called *Borel subsets of X*.

Lemma 518 *(some Borel subsets).*
Let $(X, \mathcal{T})$ be a topological space. Then, open and closed subsets are Borel subsets of X.

Moreover, if the space X is separable, then countable subsets of X are Borel subsets of X.

Proof. From Definition 517 (*Borel σ-algebra*), and Definition 474 (*σ-algebra*), the Borel subsets are the elements of the Borel σ-algebra.

Let Y be an open subset of X. Then, from Definition 517 (*Borel σ-algebra*), and Lemma 483 (*generated σ-algebra is minimum*), we have $Y \in \mathcal{T} \subset \mathcal{B}(X)$.

Let Y be a closed subset of X. Then, from Definition 249 (*topological space*, closed subset), Definition 517 (*Borel σ-algebra*), and Lemma 483 (*generated σ-algebra is minimum*), we have $Y^c \in \mathcal{T} \subset \mathcal{B}(X)$. Hence, from Definition 474 (*σ-algebra*, closedness under complement), and from **involutiveness of complement**, we have $Y = (Y^c)^c \in \mathcal{B}(X)$.

Let Y be a countable subset of X. Then, from **the definition of countability**, there exists $I \subset \mathbb{N}$, and $(x_i)_{i\in I} \in X$ such that $Y = \bigcup_{i\in I}\{x_i\}$. Let $i \in I$. Then, from **closedness of singletons in a separable space**, $\{x_i\}$ is closed, thus it belongs to $\mathcal{B}(X)$. Hence, from Definition 474 (*σ-algebra*, closedness under countable union), we have $Y \in \mathcal{B}(X)$.

Therefore, open, closed, and countable subsets are Borel subsets of X. □

Lemma 519 *(countable Borel σ-algebra generator).*
Let $(X, \mathcal{T})$ be a topological space. Let $G \subset \mathcal{T}$. Assume that all open subsets of X are countable unions of elements of G. Then, we have $\mathcal{B}(X) = \Sigma_X(G)$.

Proof. Direct consequence of Definition 517 (*Borel σ-algebra*, $\underline{\mathcal{B}(X) = \Sigma_X(\mathcal{T})}$), and Lemma 503 (*countable σ-algebra generator*, $\underline{\text{with } G_1 \stackrel{\text{def.}}{=} G \text{ and } G_2 \stackrel{\text{def.}}{=} \mathcal{T}}$). □

Remark 520. The previous lemma means that if G contains a countable topological basis of $(X, \mathcal{T})$ (and possibly other open subsets) (see Definitions 254 and 262), then the Borel σ-algebra of X is generated by G.

Remark 521. Note that metric spaces are separable topological spaces, thus both previous lemmas apply in metric spaces.

9.2 Measurable function

Definition 522 *(measurable function).* Let (X, Σ) and (X', Σ') be measurable spaces. A function $f : X \to X'$ is said *measurable (for* Σ *and* Σ'*)* iff

$$f^{-1}(\Sigma') \subset \Sigma \qquad \text{i.e. } \forall A' \in \Sigma',\ f^{-1}(A') \in \Sigma. \tag{9.1}$$

If Σ and Σ' are the Borel σ-algebras of X and X', then f is called *Borel function.*

Lemma 523 *(inverse σ-algebra).* *Let X be a set. Let (X', Σ') be a measurable space. Let $f : X \to X'$. Then, $\Sigma = f^{-1}(\Sigma')$ is the smallest σ-algebra of $f^{-1}(X') \subset X$ such that f is measurable for Σ and Σ'.*

It is called the inverse σ-algebra of Σ' by f*.*

Proof. Direct consequence of Definition 516 (*measurable space*), Definition 474 (*σ-algebra*), **homogeneity of inverse image**, **compatibility of inverse image with complement, union and intersection**, and Definition 522 (*measurable function*). □

Lemma 524 *(image σ-algebra).* *Let (X, Σ) be a measurable space. Let X' be a set. Let $f : X \to X'$. Then, $\Sigma' = \Sigma_f \stackrel{\text{def.}}{=} \{A' \subset X' \,|\, f^{-1}(A') \in \Sigma\}$ is the largest σ-algebra of $f(X)$ such that f is measurable for Σ and Σ'. It is called the* image σ-algebra of Σ by f*.*

Proof. Direct consequence of Definition 516 (*measurable space*), Definition 474 (*σ-algebra*), **compatibility of inverse image with complement, union and intersection**, **homogeneity of inverse image ($f^{-1}(\varnothing) = \varnothing$)**, and Definition 522 (*measurable function*). □

Lemma 525 *(identity function is measurable).* *Let (X, Σ) be a measurable space. Then, the identity function is measurable for Σ (and Σ).*

Proof. Direct consequence of **the definition of the identity function**, and Definition 522 (*measurable function*). □

Lemma 526 *(constant function is measurable).* *Let (X, Σ) and (X', Σ') be measurable spaces. Let $f : X \to X'$. Assume that f is constant. Then, f is measurable for Σ and Σ'.*

Proof. Let $c' \in X'$ be the constant value taken by the function f. Let $A' \in \Sigma'$.
Case $c' \in A'$. Then, we have $f^{-1}(A') = X$. **Case $c' \notin A'$.** Then, we have $f^{-1}(A') = \varnothing$. Thus, from Lemma 475 (*equivalent definition of σ-algebra*), $f^{-1}(A')$ always belongs to the σ-algebra Σ. Therefore, f is measurable for Σ and Σ'. □

Lemma 527 *(inverse image of generating family).* *Let (X, Σ) and (X', Σ') be measurable spaces. Let $f : X \to X'$. Then, we have*

$$\forall G' \subset \mathcal{P}(X'), \quad \Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G')). \tag{9.2}$$

Proof. **"Left" included in "right".** Let $G' \subset \mathcal{P}(X')$. Then, from Lemma 483 (*generated σ-algebra is minimum*), we have $G' \subset \Sigma_{X'}(G')$. Thus, from **monotonicity of inverse image**, $f^{-1}(G')$ is a subset of $f^{-1}(\Sigma_{X'}(G'))$. Hence, from Lemma 484 (*σ-algebra generation is monotone*), Lemma 523 (*inverse σ-algebra*, $\underline{f^{-1}(\Sigma_{X'}(G'))}$ is σ-algebra), and Lemma 485 (*σ-algebra generation is idempotent*), we have $\Sigma_X(f^{-1}(G')) \subset \Sigma_X(f^{-1}(\Sigma_{X'}(G'))) = f^{-1}(\Sigma_{X'}(G'))$.

"Right" included in "left". Conversely, let $A' \in G' \subset \mathcal{P}(X')$. Then, from **monotonicity of inverse image**, and Lemma 483 (*generated σ-algebra is minimum*), we have

$$f^{-1}(A') \in f^{-1}(G') \subset \Sigma_X(f^{-1}(G')).$$

Thus, from Lemma 524 (*image σ-algebra*), we have $G' \subset (\Sigma_X(f^{-1}(G')))_f$, and from Lemma 483 (*generated σ-algebra is minimum*), we have $\Sigma_{X'}(G') \subset (\Sigma_X(f^{-1}(G')))_f$. Hence, from Lemma 524 (*image σ-algebra*), and **the definition of inverse image**, we have

$$f^{-1}(\Sigma_{X'}(G')) \subset \Sigma_X(f^{-1}(G')).$$

Therefore, we have $\Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G'))$. □

Lemma 528 ***(equivalent definition of measurable function).*** *Let (X, Σ) and (X', Σ') be measurable spaces. Let $f : X \to X'$. Then, f is measurable for Σ and Σ' iff*

$$\exists G' \subset \mathcal{P}(X'), \quad \Sigma_{X'}(G') = \Sigma' \quad \Longrightarrow \quad f^{-1}(G') \subset \Sigma. \tag{9.3}$$

Proof. **"Left" implies "right".** Assume first that f is measurable for Σ and Σ'. Let $G' \stackrel{\text{def.}}{=} \Sigma'$. Then, from Lemma 485 (*σ-algebra generation is idempotent*), we have $\Sigma_{X'}(\Sigma') = \Sigma'$. Hence, from Definition 522 (*measurable function*), we have $f^{-1}(G') = f^{-1}(\Sigma') \subset \Sigma$.

"Right" implies "left". Conversely, assume now that there exists $G' \subset \mathcal{P}(X')$ such that $\Sigma_{X'}(G') = \Sigma'$ and $f^{-1}(G') \subset \Sigma$. Then, from Lemma 527 (*inverse image of generating family*), Lemma 484 (*σ-algebra generation is monotone*), and Lemma 485 (*σ-algebra generation is idempotent*), we have $f^{-1}(\Sigma') = f^{-1}(\Sigma_{X'}(G')) = \Sigma_X(f^{-1}(G')) \subset \Sigma_X(\Sigma) = \Sigma$.

Therefore, we have the equivalence. □

Lemma 529 ***(continuous is measurable).***
Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. Assume that X and X' are equipped with their Borel σ-algebras. Let $f : X \to X'$. Assume that f is continuous. Then, f is a Borel function.

Proof. Let $O' \in \mathcal{T}'$. Then, from **the definition of continuity**, $f^{-1}(O') \in \mathcal{T}$. Thus, from Lemma 483 (*generated σ-algebra is minimum*), and Definition 517 (*Borel σ-algebra*), we have $\mathcal{B}(X') = \Sigma_{X'}(\mathcal{T}')$ and $f^{-1}(\mathcal{T}') \subset \mathcal{T} \subset \Sigma_X(\mathcal{T}) = \mathcal{B}(X)$. Therefore, from Lemma 528 (*equivalent definition of measurable function*), and Definition 522 (*measurable function*), f is a Borel function from X to X'. □

Lemma 530 ***(compatibility of measurability with composition).***
Let (X, Σ), (X', Σ'), and (X'', Σ'') be measurable spaces. Let $f : X \to X'$ and $g : X' \to X''$. Assume that f is measurable for Σ and Σ', and that g is measurable for Σ' and Σ''. Then, $g \circ f$ is measurable for Σ and Σ''.

Proof. Direct consequence of **the definition of composition**, Definition 522 (*measurable function*, with f and g), and **monotonicity of inverse image**. □

9.3 Measurable subspace

Remark 531. We recall that $\sqcap$ denotes a set of traces of subsets, see Definition 216.

Lemma 532 ***(trace σ-algebra).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Let i be the canonical injection from Y to X. Then, $\Sigma \sqcap Y$ is a σ-algebra of Y, and i is measurable for $\Sigma \sqcap Y$ and Σ.

$\Sigma \sqcap Y$ *is called* trace σ-algebra of Σ on Y. *The measurable space* $(Y, \Sigma \sqcap Y)$ *is said* measurable subspace of (X, Σ).

Proof. From Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $\varnothing \in \Sigma$), and since $\varnothing$ **is absorbing for intersection**, we have $\varnothing = \varnothing \cap Y \in \Sigma \sqcap Y$.

Let $A \in \Sigma \sqcap Y$. Then, from Definition 216 (*trace of subsets of parties*), let $B \in \Sigma$ such that $A = B \cap Y$. Then, from **the definition of set difference**, **De Morgan's laws**, **distributivity of intersection over union**, **the definition of complement**, and **commutativity of intersection**, we have

$$Y \setminus A = Y \setminus (B \cap Y) = Y \cap (B \cap Y)^c = Y \cap (B^c \cup Y^c) = (Y \cap B^c) \cup (Y \cap Y^c) = B^c \cap Y.$$

Hence, from Definition 474 (*σ-algebra*, closedness under complement), we have $Y \setminus A \in \Sigma \sqcap Y$.

Let I be a set. Let $(A_i)_{i \in I} \in \Sigma \sqcap Y$. Then, from Definition 216 (*trace of subsets of parties*), for all $i \in I$, let $B_i \in \Sigma$ such that $A_i = B_i \cap Y$. Then, from **distributivity of intersection over union**, we have

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} (B_i \cap Y) = \bigcup_{i \in I} B_i \cap Y.$$

Hence, from Definition 474 (*σ-algebra*, closedness under union), we have $\bigcup_{i \in I} A_i \in \Sigma \sqcap Y$.

Therefore, from Definition 474 (*σ-algebra*), $\Sigma \sqcap Y$ is a σ-algebra of Y. Moreover, from **the definition of the canonical injection**, and Definition 522 (*measurable function*), i is measurable for $\Sigma \sqcap Y$ and Σ. □

Lemma 533 ***(measurability of measurable subspace).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Then, we have $Y \in \Sigma$ iff $\Sigma \sqcap Y = \{A \in \Sigma \,|\, A \subset Y\}$.

Proof. Let $A \in \Sigma$ such that $A \subset Y$. Then, from Definition 216 (*trace of subsets of parties*), $A = A \cap Y$ belongs to $\Sigma \sqcap Y$. Thus, we always have $\{A \in \Sigma \,|\, A \subset Y\} \subset \Sigma \sqcap Y$.

"Left" implies "right". Assume first that $Y \in \Sigma$. Let $A \in \Sigma$, and let $A' \overset{\text{def.}}{=} A \cap Y$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have $A' \in \Sigma$ and $A' \subset Y$. Thus, from Definition 216 (*trace of subsets of parties*), $\Sigma \sqcap Y$ is included in $\{A \in \Sigma \,|\, A \subset Y\}$. Hence, $\Sigma \sqcap Y = \{A \in \Sigma \,|\, A \subset Y\}$.

"Right" implies "left". Conversely, assume now that $\Sigma \sqcap Y = \{A \in \Sigma \,|\, A \subset Y\}$. Then, from Lemma 532 (*trace σ-algebra*), and Lemma 475 (*equivalent definition of σ-algebra*, contains full set), we have $Y \in \Sigma \sqcap Y \subset \Sigma$.

Therefore, we have the equivalence. □

Lemma 534 ***(generating measurable subspace).***
Let X be a set. Let $Y \subset X$. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_Y(G \sqcap Y) = \Sigma_X(G) \sqcap Y$.

Proof. Let i be the canonical injection from Y to X. Then, from Definition 216 (*trace of subsets of parties*), we have $i^{-1}(G) = G \sqcap Y$ and $i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \sqcap Y$. Therefore, from Lemma 527 (*inverse image of generating family*), we have

$$\Sigma_Y(G \sqcap Y) = \Sigma_Y(i^{-1}(G)) = i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \sqcap Y.$$

□

Lemma 535 ***(Borel sub-σ-algebra).*** *Let $(X, \mathcal{T})$ be a topological space. Let $Y \subset X$. Then, we have $\mathcal{B}(Y) = \mathcal{B}(X) \overline{\cap} Y$, and $Y \in \mathcal{B}(X)$ iff $\mathcal{B}(Y) = \{A \in \mathcal{B}(X) \,|\, A \subset Y\}$.*

Proof. Direct consequence of Definition 517 (*Borel σ-algebra*), **the definition of trace topology on Y**, Lemma 534 (*generating measurable subspace*), and Lemma 533 (*measurability of measurable subspace*). □

Lemma 536 ***(characterization of Borel subsets).*** *Let $(X, \mathcal{T})$ be a topology space. Let $(Y_n)_{n\in\mathbb{N}} \in \mathcal{B}(X)$. Assume that $X = \biguplus_{n\in\mathbb{N}} Y_n$. Let $A \subset X$. Then, we have*

$$A \in \mathcal{B}(X) \iff \forall n \in \mathbb{N}, \quad A \cap Y_n \in \mathcal{B}(Y_n). \tag{9.4}$$

Proof. **"Left" implies "right".** Assume first that $A \in \mathcal{B}(X)$. Let $n \in \mathbb{N}$. Then, $A \cap Y_n \subset Y_n$, and from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have $A \cap Y_n \in \mathcal{B}(X)$. Hence, from Lemma 535 (*Borel sub-σ-algebra*), we have $A \cap Y_n \in \mathcal{B}(Y_n)$.

"Right" implies "left". Conversely, assume now that for all $n \in \mathbb{N}$, $A \cap Y_n \in \mathcal{B}(Y_n)$. Let $n \in \mathbb{N}$. Then, since $Y_n \in \mathcal{B}(X)$, from Lemma 535 (*Borel sub-σ-algebra*), we have $A \cap Y_n \in \mathcal{B}(X)$. Hence, from Lemma 209 (*compatibility of pseudopartition with intersection*), Definition 517 (*Borel σ-algebra*), and Definition 474 (*σ-algebra*, closedness under countable union), we have

$$A = \biguplus_{n\in\mathbb{N}} (A \cap Y_n) \in \mathcal{B}(X).$$

Therefore, we have the equivalence. □

Lemma 537 ***(source restriction of measurable function).*** *Let (X, Σ) and (X', Σ') be measurable spaces. Let $f : X \to X'$. Let $Y \subset X$. Let $f_{|Y}$ be the restriction of f to the source Y. Assume that f is measurable for Σ and Σ'. Then, $f_{|Y}$ is measurable for $\Sigma \overline{\cap} Y$ and Σ'.*

Proof. Let i be the canonical injection from Y to X. Then, from Definition 216 (*trace of subsets of parties*), and Lemma 532 (*trace σ-algebra*), $i^{-1}(\Sigma) = \Sigma \overline{\cap} Y$ is a σ-algebra of Y. Thus, from Definition 522 (*measurable function*), i is measurable for $\Sigma \overline{\cap} Y$ and Σ. Therefore, from Lemma 530 (*compatibility of measurability with composition*), $f_{|Y} = f \circ i$ is measurable for $\Sigma \overline{\cap} Y$ and Σ'. □

Lemma 538 ***(destination restriction of measurable function).*** *Let (X, Σ) and (X', Σ') be measurable spaces. Let $f : X \to X'$. Let $Y' \subset X'$. Assume that $f(X) \subset Y'$. Let $f^{|Y'}$ be the restriction of f to the destination Y'. Then, f is measurable for Σ and Σ' iff $f^{|Y'}$ is measurable for Σ and $\Sigma' \overline{\cap} Y'$.*

Proof. Let $A' \in \Sigma'$. Then, from **the definition of set difference**, and **monotonicity of inverse image and complement**, we have

$$f^{-1}(A' \setminus Y') = f^{-1}(A' \cap Y'^c) \subset f^{-1}(Y'^c) \subset f^{-1}(f(X)^c) = \varnothing.$$

Then, from **compatibility of inverse image with disjoint union**, and **the definition of destination restriction**, we have

$$f^{-1}(A') = f^{-1}((A' \cap Y') \uplus (A' \setminus Y')) = f^{-1}(A' \cap Y') = \left(f^{|Y'}\right)^{-1}(A' \cap Y').$$

Hence, $f^{-1}(\Sigma') = \left(f^{|Y'}\right)^{-1}(\Sigma' \overline{\cap} Y')$. Therefore, from Definition 522 (*measurable function*), $f^{|Y'}$ is measurable for Σ and $\Sigma' \overline{\cap} Y'$. □

Lemma 539 ***(measurability of function defined on a pseudopartition).*** *Let (X, Σ) and (X', Σ') be measurable spaces. Let $I \subset \mathbb{N}$. For all $i \in I$, let $X_i \in \Sigma$ and $f_i : X \to X'$. Assume that for all $i \in I$, f_i is measurable for Σ and Σ', and that $X = \biguplus_{i \in I} X_i$. Then, the function defined by f_i on X_i for all $i \in I$ is measurable for Σ and Σ'.*

Proof. Let f be the function defined by for all $i \in I$, for all $x \in X_i$, $f(x) \stackrel{\text{def.}}{=} f_i(x)$. Let $A' \in \Sigma'$. Let $i \in I$. Then, from Definition 522 (*measurable function*, with f_i), we have $f_i^{-1}(A') \in \Sigma$. Thus, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection and union), we have $f^{-1}(A') = \bigcup_{i \in I} \left(X_i \cap f_i^{-1}(A') \right) \in \Sigma$. Hence, $f^{-1}(\Sigma') \subset \Sigma$.

Therefore, from Definition 522 (*measurable function*), f is measurable for Σ and Σ'. □

9.4 Product of measurable spaces

Remark 540. We recall that $[n..p]$ denotes an interval of integers, and that $\overline{\prod}$ and $\overline{\times}$ denote a set of Cartesian products of subsets, see Definition 217.
The concepts presented in this section are used mainly in Section 13.4.

Definition 541 ***(tensor product of σ-algebras).***
Let $m \in [2..\infty)$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space. Let $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$ and $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in[1..m]}\Sigma_i$. The *(tensor) product of the σ-algebras* $(\Sigma_i)_{i\in[1..m]}$ on X is the σ-algebra generated by $\overline{\Sigma}$; it is denoted $\bigotimes_{i\in[1..m]} \Sigma_i \stackrel{\text{def.}}{=} \Sigma_X(\overline{\Sigma})$.

Lemma 542 ***(product of measurable subsets is measurable).***
Let $m \in [2..\infty)$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space, and $A_i \in \Sigma_i$. Let $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in[1..m]}\Sigma_i$ and $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_i$. Then, $\prod_{i\in[1..m]} A_i \in \overline{\Sigma} \subset \Sigma$.

Proof. Direct consequence of Definition 217 (*product of subsets of parties*), Definition 541 (*tensor product of σ-algebras*), and Lemma 483 (*generated σ-algebra is minimum*, $\underline{\overline{\Sigma} \subset \Sigma_X(\overline{\Sigma})}$). □

Lemma 543 ***(measurability of function to product space).***
Let $m \in [2..\infty)$. Let (X, Σ) be a measurable space. For all $i \in [1..m]$, let (X'_i, Σ'_i) be a measurable space, and $f_i : X \to X'_i$. Then, $(f_i)_{i\in[1..m]}$ is measurable for Σ and $\bigotimes_{i\in[1..m]} \Sigma'_i$ iff for all $i \in [1..m]$, f_i is measurable for Σ and Σ'_i.

Proof. Let $\Sigma' \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma'_i$.

"Left" implies "right". Assume first that $(f_i)_{i\in[1..m]}$ is measurable for Σ and Σ'.

Let $i \in [1..m]$. Let $A'_i \in \Sigma'_i$. Let $A' \stackrel{\text{def.}}{=} X'_1 \times \ldots \times A'_i \times \ldots \times X'_m$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, $\underline{X'_i \in \Sigma'_i}$), Definition 541 (*tensor product of σ-algebras*), and Lemma 542 (*product of measurable subsets is measurable*), we have $A' \in \Sigma'$. Thus, from Definition 522 (*measurable function*, $\underline{\text{with } f}$), we have $f_i^{-1}(A'_i) = f^{-1}(A') \in \Sigma$. Hence, from Definition 522 (*measurable function*), f_i is measurable for Σ and Σ'_i.

"Right" implies "left". Assume now that for all $i \in [1..m]$, f_i is measurable for Σ and Σ'_i.
Let $\overline{\Sigma}' \stackrel{\text{def.}}{=} \overline{\prod}_{i\in[1..m]}\Sigma'_i$ and $f \stackrel{\text{def.}}{=} (f_i)_{i\in[1..p]}$. Let $A' \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} A'_i \in \overline{\Sigma}'$. From Definition 217 (*product of subsets of parties*), for all $i \in [1..m]$, we have $A'_i \in \Sigma'_i$. Let $x \in X$. Then, $f(x) \in A'$ iff for all $i \in [1..m]$, $f_i(x) \in A'_i$. Thus, we have $f^{-1}(A') = \bigcap_{i\in[1..m]} f_i^{-1}(A'_i)$. Hence, from Definition 522 (*measurable function*, $\underline{\text{with } f_i}$), and Lemma 475 (*equivalent definition of σ-algebra*, $\underline{\text{closedness under countable intersection}}$), we have $f^{-1}(\overline{\Sigma}') = \bigcap_{i\in[1..m]} f_i^{-1}(\Sigma'_i) \subset \Sigma$. Thus, from Lemma 528 (*equivalent definition of measurable function*, $\underline{\text{with } G' \stackrel{\text{def.}}{=} \overline{\Sigma}'}$), f is measurable for Σ and Σ'.

Therefore, we have the equivalence. □

Lemma 544 ***(canonical projection is measurable).***
*Let $m \in [2..\infty)$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space, and let π_i be the canonical projection from $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$ onto X_i. Let $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_i$.
Then, Σ is the smallest σ-algebra on X such that for all $i \in [1..m]$, π_i is measurable for Σ and Σ_i.*

Proof. Let Σ' be a σ-algebra on X. Then, from Lemma 543 (*measurability of function to product space*, $\underline{\text{with } \Sigma \stackrel{\text{def.}}{=} \Sigma',\ X'_i \stackrel{\text{def.}}{=} X_i,\ \Sigma'_i \stackrel{\text{def.}}{=} \Sigma_i, \text{ and } f \stackrel{\text{def.}}{=} \mathrm{Id}_X = (\pi_i)_{i\in[1..m]}}$), Definition 522 (*measur-*

able function), and **reflexivity of inclusion** we have

$$\begin{aligned} & \forall i \in [1..m],\ \pi_i \text{ is measurable for } \Sigma' \text{ and } \Sigma_i \\ \iff & \quad \mathrm{Id}_X \text{ is measurable for } \Sigma' \text{ and } \Sigma \\ \iff & \quad \mathrm{Id}_X^{-1}(\Sigma) = \Sigma \subset \Sigma'. \end{aligned}$$

□

Lemma 545 ***(permutation is measurable).*** *Let $m \in [2..\infty)$. Let ψ be a permutation of $[1..m]$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space, and let π_i be the canonical projection from $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$ onto X_i. Let $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_i$, and $\Sigma^\psi \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_{\psi(i)}$. Then, the permutation of coordinates $(\pi_{\psi(i)})_{i\in[1..m]}$ is measurable for Σ and Σ^ψ.*

Proof. Direct consequence of Lemma 543 (*measurability of function to product space*), and Lemma 544 (*canonical projection is measurable*, with $X'_i \stackrel{\text{def.}}{=} X_{\psi(i)}$ and $f_i \stackrel{\text{def.}}{=} \pi_{\psi(i)}$). □

Lemma 546 ***(generating product measurable space).*** *Let $m \in [2..\infty)$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space, $G_i \subset \mathcal{P}(X_i)$, and assume that $X_i \in G_i$ and $\Sigma_i = \Sigma_{X_i}(G_i)$. Let $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$ and $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_i$. Then, we have $\Sigma = \Sigma_X \left(\prod_{i\in[1..m]} G_i\right)$.*

Proof. Let $\overline{G} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in[1..m]} G_i$.

$\boldsymbol{\Sigma \subset \Sigma_X(\overline{G})}$**.** Let $i \in [1..m]$. Let π_i be the canonical projection from X onto X_i. Then, from Lemma 527 (*inverse image of generating family*), we have

$$\pi_i^{-1}(\Sigma_i) = \pi_i^{-1}(\Sigma_{X_i}(G_i)) = \Sigma_X(\pi_i^{-1}(G_i)).$$

Moreover, since $X_i \in G_i$, and from Definition 217 (*product of subsets of parties*), we have

$$\pi_i^{-1}(G_i) = \{X_1 \times \ldots \times A_i \times \ldots \times X_m \mid A_i \in G_i\} \subset \overline{G}.$$

Then, from Lemma 484 (*σ-algebra generation is monotone*), we have $\pi_i^{-1}(\Sigma_i) \subset \Sigma_X(\overline{G})$. Thus, from Definition 522 (*measurable function*), π_i is measurable for $\Sigma_X(\overline{G})$ and Σ_i. Hence, from Lemma 544 (*canonical projection is measurable*, smallest σ-algebra), we have $\Sigma \subset \Sigma_X(\overline{G})$.

$\boldsymbol{\Sigma_X(\overline{G}) \subset \Sigma}$**.** Let $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i\in[1..m]} \Sigma_i$.
Direct consequence of Lemma 483 (*generated σ-algebra is minimum*, $G_i \subset \Sigma_i$), Definition 217 (*product of subsets of parties*, $\overline{G} \subset \overline{\Sigma}$), Lemma 484 (*$\sigma$-algebra generation is monotone*, $\Sigma_X(\overline{G})$ is included in $\Sigma_X(\overline{\Sigma})$), and Definition 541 (*tensor product of σ-algebras*, $\Sigma = \Sigma_X(\overline{\Sigma})$),

Therefore, we have the equality. □

Remark 547. For the sake of simplicity, we only present the remainder of this section in the case of the product of two measure spaces. When $i \in \{1, 2\}$, the complement $\{1, 2\} \setminus \{i\}$ is $\{3 - i\}$.

Definition 548 ***(section in Cartesian product).*** Let X_1 and X_2 be sets. Let $A \subset X_1 \times X_2$. Let $i \in \{1, 2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $x_i \in X_i$. The *i-th section of A in x_i* is the subset

$$s_i(x_i, A) \stackrel{\text{def.}}{=} \{x_j \in X_j \mid (x_1, x_2) \in A\}. \tag{9.5}$$

Lemma 549 ***(section of product).*** *Let X_1 and X_2 be sets. Let $A_1 \in X_1$ and $A_2 \in X_2$. Let $i \in \{1, 2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $x_i \in X_i$. Then, we have*

$$s_i(x_i, A_1 \times A_2) = \begin{cases} A_j & \textit{when } x_i \in A_i, \\ \varnothing & \textit{otherwise.} \end{cases} \tag{9.6}$$

Proof. Direct consequence of Definition 548 (*section in Cartesian product*). □

Lemma 550 ***(compatibility of section with set operations).*** *Let X_1 and X_2 be sets. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$. Let $i \in \{1,2\}$. Let $x_i \in X_i$. Then, the sections are compatible with the empty set, the complement, countable union and intersection, and are monotone:*

$$s_i(x_i, \varnothing) = \varnothing, \tag{9.7}$$

$$\forall A \subset X, \quad s_i(x_i, A^c) = s_i(x_i, A)^c, \tag{9.8}$$

$$\forall I \subset \mathbb{N},\ \forall (A_n)_{n\in I} \subset X, \quad s_i\left(x_i, \bigcup_{n\in I} A_n\right) = \bigcup_{n\in I} s_i(x_i, A_n), \tag{9.9}$$

$$\forall I \subset \mathbb{N},\ \forall (A_n)_{n\in I} \subset X, \quad s_i\left(x_i, \bigcap_{n\in I} A_n\right) = \bigcap_{n\in I} s_i(x_i, A_n), \tag{9.10}$$

$$\forall A, B \subset X, \quad A \subset B \implies s_i(x_i, A) \subset s_i(x_i, B). \tag{9.11}$$

Proof. Direct consequences of Definition 548 (*section in Cartesian product*). □

Lemma 551 ***(measurability of section).*** *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. Let $A \in \Sigma_1 \otimes \Sigma_2$. Let $i \in \{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $x_i \in X_i$. Then, we have $s_i(x_i, A) \in \Sigma_j$.*

Proof. Let $\overline{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. Let $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \subset X \mid s_i(x_i, A) \in \Sigma_j\}$.

From Definition 516 (*measurable space*), Σ_j is a σ-algebras.

Let $A_1 \in \Sigma_1$, and $A_2 \in \Sigma_2$. Then, from Lemma 549 (*section of product*, $s_i(x_i, A_1 \times A_2)$ belongs to $\{\varnothing, A_j\}$), Definition 474 (*σ-algebra*, $\varnothing, A_j \in \Sigma_j$), and the definition of $\mathcal{S}_i$, we have $A_1 \times A_2 \in \mathcal{S}_i$. Hence, from Definition 217 (*product of subsets of parties*), we have $\overline{\Sigma} \subset \mathcal{S}_i$.

From Lemma 550 (*compatibility of section with set operations*), and Definition 474 (*σ-algebra*), $\mathcal{S}_i$ contains $\varnothing$, and is closed under complement and countable union. Thus, from Definition 474 (*σ-algebra*), $\mathcal{S}_i$ is a σ-algebra on X. Hence, from Definition 541 (*tensor product of σ-algebras*, Σ is a generated by $\overline{\Sigma}$), Lemma 483 (*generated σ-algebra is minimum*), we have $\Sigma \subset \mathcal{S}_i$.

Therefore, for all $A \in \Sigma$ we have $s_i(x_i, A) \in \Sigma_j$. □

Lemma 552 ***(countable union of sections is measurable).*** *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. Let $I \subset \mathbb{N}$. Let $(A_n)_{n\in I} \in \Sigma_1 \otimes \Sigma_2$. Let i be in $\{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $x_i \in X_i$. Then, we have*

$$s_i\left(x_i, \bigcup_{n\in I} A_n\right) = \bigcup_{n\in I} s_i(x_i, A_n) \in \Sigma_j. \tag{9.12}$$

Proof. Direct consequence of Lemma 550 (*compatibility of section with set operations*, with countable union), Definition 516 (*measurable space*, Σ_j is a σ-algebra), Definition 474 (*σ-algebra*, closedness under countable union), and Lemma 551 (*measurability of section*). □

Lemma 553 ***(countable intersection of sections is measurable).*** *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. Let $(A_n)_{n\in\mathbb{N}} \in \Sigma_1 \otimes \Sigma_2$. Let i be in $\{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $x_i \in X_i$. Then, we have*

$$s_i\left(x_i, \bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} s_i(x_i, A_n) \in \Sigma_j. \tag{9.13}$$

Proof. Direct consequence of Lemma 550 (*compatibility of section with set operations*, with countable intersection), Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), and Lemma 551 (*measurability of section*). □

Lemma 554 ***(indicator of section).*** *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. Let $A \subset X_1 \times X_2$. Let $i \in \{1, 2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Then, we have*

$$\forall x_1 \in X_1, \quad \forall x_2 \in X_2, \quad \mathbb{1}_A(x_1, x_2) = \mathbb{1}_{s_i(x_i, A)}(x_j). \tag{9.14}$$

Proof. Direct consequence of Definition 548 (*section in Cartesian product*), and **the definition of the indicator function**. □

Lemma 555 ***(measurability of function from product space).*** *Let (X_1, Σ_1), (X_2, Σ_2) and (X', Σ') be measurable spaces. Let $i \in \{1, 2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Let $f : X_1 \times X_2 \to X'$. Assume that f is measurable for $\Sigma_1 \otimes \Sigma_2$ and Σ'. Then, for all $x_i \in X_i$, the function $(x_j \mapsto f(x_1, x_2))$ is measurable for Σ_j and Σ'.*

Proof. Direct consequence of **properties of inverse image $((f^j_{x_i})^{-1}(A') = s_i(x_i, f^{-1}(A'))$ where $f^j_{x_i} \stackrel{\text{def.}}{=} (x_j \mapsto f(x_1, x_2))$ and $A' \in \Sigma')$**, Lemma 551 (*measurability of section*, $\underline{(f^j_{x_i})^{-1}(A') \in \Sigma_j}$), and Definition 522 (*measurable function*). □

Remark 556. Note that the reciprocal of the previous lemma is false.

Indeed, let us build a counter-example. Let $X_1 = X_2 = X' \stackrel{\text{def.}}{=} \mathbb{R}$, $\Sigma_1 = \Sigma_2 \stackrel{\text{def.}}{=} \Sigma_{\mathbb{R}}(\{\{x\}\}_{x \in \mathbb{R}})$, and $\Sigma' \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$. Let $D \stackrel{\text{def.}}{=} \{(x, x) \,|\, x \in \mathbb{R}\} \subset X_1 \times X_2$. Then, the tensor product σ-algebra $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ is also generated by all singletons of $\mathbb{R}^2$, and D is not measurable. Let $f = \mathbb{1}_D$. Then, f is not measurable for Σ and Σ', but for all $x \in X$, $(y \mapsto f(x, y)) = (y \mapsto f(y, x)) = \mathbb{1}_{\{x\}}$ is obviously measurable for Σ and Σ'.

Chapter 10

Measurability and numbers

Contents

Remark 557. We address in this chapter the specific case of the measurable spaces of numbers $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and of measurable functions to them.

10.1 Borel subset of numbers

10.1.1 Borel subset of real numbers

Lemma 558 ***(Borel σ-algebra of $\mathbb{R}$).***
$\mathcal{B}(\mathbb{R})$ is generated by any of the following sets of intervals:

$$\{(a,b)\}_{a<b}, \quad \{[a,b)\}_{a<b}, \quad \{[a,b]\}_{a<b}, \quad \{(a,b]\}_{a<b}, \tag{10.1}$$

$$\{(-\infty,b)\}_{b\in\mathbb{R}}, \quad \{(-\infty,b]\}_{b\in\mathbb{R}}, \quad \{(a,\infty)\}_{a\in\mathbb{R}}, \quad \{[a,\infty)\}_{a\in\mathbb{R}}. \tag{10.2}$$

Proof. From Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$), and Lemma 519 (*countable Borel σ-algebra generator*), we have $\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a,b)\}_{a<b})$.

Let $a, b \in \mathbb{R}$ such that $a < b$. Then, we have the following countable unions:

$$[a,b) = \{a\} \cup (a,b), \quad [a,b] = \{a\} \cup (a,b) \cup \{b\}, \quad (a,b] = (a,b) \cup \{b\},$$
$$(a,b) = \bigcup_{n\in\mathbb{N}} \left[a + \tfrac{1}{n+1}, b\right) = \bigcup_{n\in\mathbb{N}} \left[a + \tfrac{1}{n+1}, b - \tfrac{1}{n+1}\right] = \bigcup_{n\in\mathbb{N}} \left(a, b - \tfrac{1}{n+1}\right].$$

Thus, from Definition 482 (*generated σ-algebra*), Definition 474 (*σ-algebra*, closedness under countable union), and Lemma 518 (*some Borel subsets*, singletons are Borel subsets), we have

$$\{[a,b)\}_{a<b},\ \{[a,b]\}_{a<b},\ \{(a,b]\}_{a<b} \ \subset\ \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a,b)\}_{a<b}),$$
$$\{(a,b)\}_{a<b} \ \subset\ \Sigma_{\mathbb{R}}(\{[a,b)\}_{a<b}),\ \Sigma_{\mathbb{R}}(\{[a,b]\}_{a<b}),\ \Sigma_{\mathbb{R}}(\{(a,b]\}_{a<b}).$$

Hence, from Lemma 501 (*other σ-algebra generator*), we have

$$\mathcal{B}(\mathbb{R}) = \Sigma_\mathbb{R}(\{(a,b)\}_{a<b}) = \Sigma_\mathbb{R}(\{[a,b)\}_{a<b}) = \Sigma_\mathbb{R}(\{[a,b]\}_{a<b}) = \Sigma_\mathbb{R}(\{(a,b]\}_{a<b}).$$

Let $a, b \in \mathbb{R}$ such that $a < b$. Then, we have the following (finite) unions and intersections:

$$(-\infty, b] = (-\infty, b) \cup \{b\}, \quad [a, \infty) = \{a\} \cup (a, \infty),$$
$$[a, b) = (-\infty, a)^c \cap (-\infty, b) = [a, \infty) \cap [b, \infty)^c,$$
$$(a, b] = (-\infty, a]^c \cap (-\infty, b] = (a, \infty) \cap (b, \infty)^c.$$

Thus, since $(-\infty, b)$ and (a, ∞) are open, and from Definition 482 (*generated σ-algebra*), Lemma 475 (*equivalent definition of σ-algebra*, closedness under complement, countable union and intersection), and Lemma 518 (*some Borel subsets*, singletons are Borel subsets), we have

$$\{(-\infty, b)\}_{b\in\mathbb{R}},\ \{[a, \infty)\}_{a\in\mathbb{R}} \ \subset\ \mathcal{B}(\mathbb{R}) = \Sigma_\mathbb{R}(\{[a,b)\}_{a<b}),$$
$$\{(-\infty, b]\}_{b\in\mathbb{R}},\ \{(a, \infty)\}_{a\in\mathbb{R}} \ \subset\ \mathcal{B}(\mathbb{R}) = \Sigma_\mathbb{R}(\{(a,b]\}_{a<b}),$$
$$\{[a,b)\}_{a<b} \ \subset\ \Sigma_\mathbb{R}(\{(-\infty, b)\}_{b\in\mathbb{R}}),\ \Sigma_\mathbb{R}(\{[a, \infty)\}_{a\in\mathbb{R}}),$$
$$\{(a,b]\}_{a<b} \ \subset\ \Sigma_\mathbb{R}(\{(-\infty, b]\}_{b\in\mathbb{R}}),\ \Sigma_\mathbb{R}(\{(a, \infty)\}_{a\in\mathbb{R}}).$$

Hence, from Lemma 501 (*other σ-algebra generator*), we have

$$\mathcal{B}(\mathbb{R}) = \Sigma_\mathbb{R}(\{[a,b)\}_{a<b}) = \Sigma_\mathbb{R}(\{(-\infty, b)\}_{b\in\mathbb{R}}) = \Sigma_\mathbb{R}(\{[a, \infty)\}_{a\in\mathbb{R}}),$$
$$\mathcal{B}(\mathbb{R}) = \Sigma_\mathbb{R}(\{(a,b]\}_{a<b}) = \Sigma_\mathbb{R}(\{(-\infty, b]\}_{b\in\mathbb{R}}) = \Sigma_\mathbb{R}(\{(a, \infty)\}_{a\in\mathbb{R}}).$$

Therefore, all eight sets of intervals generate the Borel σ-algebra of $\mathbb{R}$. □

Lemma 559 *(countable generator of Borel σ-algebra of $\mathbb{R}$).*
*In Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*), finite bounds of intervals, a and b, may be taken rational.*

Proof. Same proof as for Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*), but with using the countability of the topological basis of $\mathbb{R}$ equipped with the Euclidean distance through Theorem 359 (*$\mathbb{R}$ is second-countable*) instead of Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*). □

10.1.2 Borel subset of extended real numbers

Lemma 560 *(Borel σ-algebra of $\overline{\mathbb{R}}$).*
$\mathcal{B}(\overline{\mathbb{R}})$ is generated by any of the following sets of intervals:

(10.3) $$\{[-\infty, b)\}_{b\in\mathbb{R}}, \quad \{[-\infty, b]\}_{b\in\mathbb{R}}, \quad \{(a, \infty]\}_{a\in\mathbb{R}}, \quad \{[a, \infty]\}_{a\in\mathbb{R}}.$$

Proof. From Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*, similar proof in $\overline{\mathbb{R}}$), and Lemma 519 (*countable Borel σ-algebra generator*), we have

$$\mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a,b)\}_{a<b} \cup \{[-\infty, b)\}_{b\in\mathbb{R}} \cup \{(a, \infty]\}_{a\in\mathbb{R}}).$$

Let $a, b \in \mathbb{R}$ such that $a < b$. Then, we have the following countable unions and (finite) intersections:

$$[-\infty, b) = \bigcup_{n\in\mathbb{N}} \left(b - \tfrac{1}{n+1}, \infty\right]^c, \quad (a, \infty] = \bigcup_{n\in\mathbb{N}} \left[-\infty, a + \tfrac{1}{n+1}\right)^c,$$
$$(a, b) = \bigcup_{n\in\mathbb{N}} \left[-\infty, a + \tfrac{1}{n+1}\right)^c \cap [-\infty, b) = (a, \infty] \cap \bigcup_{n\in\mathbb{N}} \left(b - \tfrac{1}{n+1}, \infty\right]^c.$$

Thus, from Definition 482 (*generated σ-algebra*), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under complement, countable union and intersection), we have

$$\{[-\infty,b)\}_{b\in\mathbb{R}},\ \{[a,\infty]\}_{a\in\mathbb{R}}\ \subset\ \mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}\left(\{(a,b)\}_{a<b}\cup\{[-\infty,b)\}_{b\in\mathbb{R}}\cup\{(a,\infty]\}_{a\in\mathbb{R}}\right),$$
$$\{(a,b)\}_{a<b},\ \{[-\infty,b)\}_{b\in\mathbb{R}},\ \{(a,\infty]\}_{a\in\mathbb{R}}\ \subset\ \Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b)\}_{b\in\mathbb{R}}),\ \Sigma_{\overline{\mathbb{R}}}(\{(a,\infty]\}_{a\in\mathbb{R}}).$$

Hence, from Lemma 501 (*other σ-algebra generator*), we have

$$\mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{(a,b)\}_{a<b}\cup\{[-\infty,b)\}_{b\in\mathbb{R}}\cup\{(a,\infty]\}_{a\in\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b)\}_{b\in\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{(a,\infty]\}_{a\in\mathbb{R}}).$$

Let $a,b\in\mathbb{R}$ such that $a<b$. Then, we have the following complements:

$$[-\infty,b]=(b,\infty]^c,\quad [a,\infty]=[-\infty,a)^c,$$
$$(a,\infty]=[-\infty,a]^c,\quad [-\infty,b)=[b,\infty]^c.$$

Thus, from Definition 482 (*generated σ-algebra*), Lemma 475 (*equivalent definition of σ-algebra*, closedness under complement, countable union and intersection), and Lemma 501 (*other σ-algebra generator*), we have

$$\{[-\infty,b]\}_{b\in\mathbb{R}}\ \subset\ \mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{(a,\infty]\}_{a\in\mathbb{R}}),$$
$$\{[a,\infty]\}_{a\in\mathbb{R}}\ \subset\ \mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b)\}_{b\in\mathbb{R}}),$$
$$\{(a,\infty]\}_{a\in\mathbb{R}}\ \subset\ \Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b]\}_{b\in\mathbb{R}}),$$
$$\{[-\infty,b)\}_{b\in\mathbb{R}}\ \subset\ \Sigma_{\overline{\mathbb{R}}}(\{[a,\infty]\}_{a\in\mathbb{R}}).$$

Hence, from Lemma 501 (*other σ-algebra generator*), we have

$$\mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{(a,\infty]\}_{a\in\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b]\}_{b\in\mathbb{R}}),$$
$$\mathcal{B}(\overline{\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{[-\infty,b)\}_{b\in\mathbb{R}})=\Sigma_{\overline{\mathbb{R}}}(\{[a,\infty]\}_{a\in\mathbb{R}}).$$

Therefore, all four sets of intervals generate the Borel σ-algebra of $\overline{\mathbb{R}}$. □

Lemma 561 ***(Borel subsets of*** $\overline{\mathbb{R}}$ ***and*** $\mathbb{R}$***).*** *Let $A\subset\overline{\mathbb{R}}$. Then, $A\in\mathcal{B}(\overline{\mathbb{R}})$ iff $A\cap\mathbb{R}\in\mathcal{B}(\mathbb{R})$.*

Proof. Direct consequence of Definition 517 (*Borel σ-algebra*, $\mathbb{R}$ (open subset) belongs to $\mathcal{B}(\overline{\mathbb{R}})$), Definition 474 (*σ-algebra*, $\{-\infty,\infty\}=\mathbb{R}^c\in\mathcal{B}(\overline{\mathbb{R}})$), Lemma 536 (*characterization of Borel subsets*, with $X\stackrel{\text{def.}}{=}\overline{\mathbb{R}}$, $n\stackrel{\text{def.}}{=}2$, $Y_1\stackrel{\text{def.}}{=}\mathbb{R}$, and $Y_2=\{-\infty,\infty\}$), Lemma 535 (*Borel sub-σ-algebra*, with $X\stackrel{\text{def.}}{=}\overline{\mathbb{R}}$ and $Y=\{-\infty,\infty\}$, i.e. $\mathcal{B}(\{-\infty,\infty\})=\mathcal{P}(\{-\infty,\infty\})$, and thus $A\cap\{-\infty,\infty\}$ always belongs to $\mathcal{B}(\{-\infty,\infty\})$). □

Remark 562. In other words, Borel subsets of $\overline{\mathbb{R}}$ are Borel subsets of $\mathbb{R}$, or the union of Borel subsets of $\mathbb{R}$ with $\{-\infty\}$, $\{\infty\}$, or $\{-\infty,\infty\}$.

10.1.3 Borel subset of nonnegative numbers

Lemma 563 ***(Borel σ-algebra of*** $\mathbb{R}_+$***).*** *We have $\mathcal{B}(\mathbb{R}_+)=\{A\in\mathcal{B}(\mathbb{R})\,|\,A\subset\mathbb{R}_+\}$.*

Proof. In $\mathbb{R}$, we have $\mathbb{R}_+=[0,\infty)=(-\infty,0)^c$. Thus, from Definition 474 (*σ-algebra*, closedness under complement), and Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$), $\mathbb{R}_+$ belongs to $\mathcal{B}(\mathbb{R})$. Therefore, from Lemma 535 (*Borel sub-σ-algebra*), we have $\mathcal{B}(\mathbb{R}_+)=\{A\in\mathcal{B}(\mathbb{R})\,|\,A\subset\mathbb{R}_+\}$. □

Lemma 564 ***(Borel σ-algebra of*** $\overline{\mathbb{R}}_+$***).***
$\mathcal{B}(\overline{\mathbb{R}}_+)$ is generated by any of the following sets of intervals:

$$\{[0,b)\}_{b\in\mathbb{R}_+},\quad \{[0,b]\}_{b\in\mathbb{R}_+},\quad \{(a,\infty]\}_{a\in\mathbb{R}_+},\quad \{[a,\infty]\}_{a\in\mathbb{R}_+}. \tag{10.4}$$

Proof. Direct consequence of Lemma 560 (*Borel σ-algebra of* $\overline{\mathbb{R}}$), and Lemma 534 (*generating measurable subspace*). □

10.1.4 Product of Borel subsets of numbers

Lemma 565 ***(Borel σ-algebra of $\mathbb{R}^n$).***
Let $n \in [2..\infty)$. For all $i \in [1..n]$, let $\mathcal{B}_i \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$. Then, we have

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^{n\otimes} \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..n]} \mathcal{B}_i. \tag{10.5}$$

Proof. Let $G \stackrel{\text{def.}}{=} \{(a,b) \,|\, a,b \in \mathbb{Q} \wedge a < b\}$, $G' \stackrel{\text{def.}}{=} G \cup \{\mathbb{R}\}$, and $\overline{G}' \stackrel{\text{def.}}{=} \prod_{i\in[1..n]} G'$. Then, from Lemma 559 (*countable generator of Borel σ-algebra of* $\mathbb{R}$), Lemma 475 (*equivalent definition of σ-algebra*, contains full set), and Lemma 502 (*complete generated σ-algebra*, with $G' \stackrel{\text{def.}}{=} \{\mathbb{R}\} \subset \mathcal{B}(\mathbb{R})$), we have

$$\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(G) = \Sigma_{\mathbb{R}}(G').$$

Hence, from Lemma 546 (*generating product measurable space*), we have $\mathcal{B}(\mathbb{R})^{n\otimes} = \Sigma_{\mathbb{R}^n}(\overline{G}')$.

Let d be the Euclidean distance on $\mathbb{R}^n$. Then, from Lemma 360 ($\mathbb{R}^n$ *is second-countable*, $\overline{G}$ countable topological basis of $(\mathbb{R}^n, d)$), and Lemma 266 (*complete countable topological basis of product space*, with $A_i \stackrel{\text{def.}}{=} \mathbb{R}$ open in $\mathbb{R}$), $\overline{G}'$ is a countable topological basis of $(\mathbb{R}^n, d)$. Hence, from Lemma 519 (*countable Borel σ-algebra generator*, with $X \stackrel{\text{def.}}{=} \mathbb{R}^n$ and $G \stackrel{\text{def.}}{=} \overline{G}'$), we have $\mathcal{B}(\mathbb{R}^n) = \Sigma_{\mathbb{R}^n}(\overline{G}')$.

Therefore, we have the equality $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^{m\otimes}$. □

Remark 566. Note that the previous lemma also holds for the product of subsets of $\mathbb{R}$. In particular, this more general result could be used to prove that $\mathcal{B}(\mathbb{R}_+^n) = (\mathcal{B}(\mathbb{R}_+))^{n\otimes}$.

Note also that identifying $\mathbb{R}^2$ with $\mathbb{C}$ via the canonical isometry $((x,y) \mapsto x+iy)$, allows to identify open subsets of $\mathbb{R}^2$ to those of $\mathbb{C}$. Hence, a function from some measurable space (X,Σ) to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is measurable iff its real and imaginary parts ($\operatorname{Re} f$ and $\operatorname{Im} f$) are measurable for Σ and $\mathcal{B}(\mathbb{R})$.

10.2 Measurable numeric function

10.2.1 Measurable numeric function to $\mathbb{R}$

Definition 567 ***($\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$).*** Let (X, Σ) be a measurable space. The set of functions $X \to \mathbb{R}$ that are measurable for Σ and $\mathcal{B}(\mathbb{R})$ is called the *vector space of finite-valued measurable functions (over X)*; it is denoted $\mathcal{M}_{\mathbb{R}}(X, \Sigma)$ (or simply $\mathcal{M}_{\mathbb{R}}$).

Remark 568. The set $\mathcal{M}_{\mathbb{R}}$ is shown below to be a vector space; hence its name.

Lemma 569 ***(measurability of indicator function).*** *Let (X, Σ) be a measurable space. Let $A \subset X$. Then, we have $\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}$ iff $A \in \Sigma$.*

Proof. $\mathbb{1}_A$ takes the values 0 and 1. Let $A' \in \mathcal{B}(\mathbb{R})$.
Case $0, 1 \in A'$. Then, $\mathbb{1}_A^{-1}(A') = X$. **Case $0 \in A'$ and $1 \notin A'$.** Then, $\mathbb{1}_A^{-1}(A') = A^c$.
Case $0 \notin A'$ and $1 \in A'$. Then, $\mathbb{1}_A^{-1}(A') = A$. **Case $0, 1 \notin A'$.** Then, $\mathbb{1}_A^{-1}(A') = \varnothing$.

Hence, from Definition 516 (*measurable space*, Σ is a σ-algebra), and Lemma 475 (*equivalent definition of σ-algebra*), $\mathbb{1}_A^{-1}(A')$ belongs to the σ-algebra Σ iff $A \in \Sigma$. Therefore, from Definition 474 (*σ-algebra*), and Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), $\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}$ iff $A \in \Sigma$. □

Lemma 570 ***(measurability of numeric function to $\mathbb{R}$).*** *Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$. Then, $f \in \mathcal{M}_{\mathbb{R}}$ iff one of the following conditions is satisfied:*

$$\forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a) \in \Sigma, \tag{10.6}$$

$$\forall a \in \mathbb{R}, \quad \{f \leqslant a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a] \in \Sigma, \tag{10.7}$$

$$\forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty) \in \Sigma, \tag{10.8}$$

$$\forall a \in \mathbb{R}, \quad \{f \geqslant a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty) \in \Sigma. \tag{10.9}$$

Proof. **"Left" implies "right".** Direct consequence of Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), Definition 522 (*measurable function*), Definition 517 (*Borel σ-algebra*), Lemma 518 (*some Borel subsets*), **open intervals of $\mathbb{R}$ are open subsets**, and **closed intervals of $\mathbb{R}$, are closed subsets**.

"Right" implies "left". Direct consequence of Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*), Lemma 528 (*equivalent definition of measurable function*), and Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*).

Therefore, we have the equivalence. □

Lemma 571 ***(inverse image is measurable in $\mathbb{R}$).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}_{\mathbb{R}}$. Then, we have*

$$\forall a \in \mathbb{R}, \quad \{f = a\} \stackrel{\text{def.}}{=} f^{-1}(a) \in \Sigma. \tag{10.10}$$

Proof. Direct consequence of Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), Definition 522 (*measurable function*), Lemma 518 (*some Borel subsets*, singletons are measurable). □

Lemma 572 ***($\mathcal{M}_{\mathbb{R}}$ is algebra).*** *Let (X, Σ) be a measurable space. Then, $\mathcal{M}_{\mathbb{R}}$ is a subalgebra of $(\mathbb{R}^X, +, \cdot, \times)$.*

Proof. Let $f, g \in \mathcal{M}_{\mathbb{R}}$. Then, from Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), Lemma 543 (*measurability of function to product space*), and Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*, with $m = 2$), the function $(f, g) : X \to \mathbb{R}^2$ is measurable for Σ and $\mathcal{B}(\mathbb{R}^2)$. Moreover, from **continuity of addition and multiplication from $\mathbb{R}^2$ to $\mathbb{R}$**, Lemma 529 (*continuous is measurable*), and Lemma 530 (*compatibility of measurability with composition*), $f + g$ and fg are measurable for Σ and $\mathcal{B}(\mathbb{R})$.

Let $f : X \to \mathbb{R}$. Let $a \in \mathbb{R}$. Then, from Lemma 526 (*constant function is measurable*), and Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), we have $a \in \mathcal{M}_{\mathbb{R}}$. Thus, from the previous result, we have $af \in \mathcal{M}_{\mathbb{R}}$.

Therefore, from Lemma 228 (*$\mathbb{K}$ is $\mathbb{K}$-algebra*, with $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$), Lemma 231 (*algebra of functions to algebra*, with $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$), and Lemma 236 (*closed under algebra operations is subalgebra*), $\mathcal{M}_{\mathbb{R}}$ is a subalgebra of $(\mathbb{R}^X, +, \cdot, \times)$. □

Remark 573. Note that we may also show that $\mathcal{M}_{\mathbb{C}}$, the space of measurable functions to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, is a subalgebra of $(\mathbb{C}, +, \cdot, \times)$.

Lemma 574 ***($\mathcal{M}_{\mathbb{R}}$ is vector space).***
Let (X, Σ) be a measurable space. Then, $\mathcal{M}_{\mathbb{R}}$ is a vector space with the zero function as zero.

Proof. Direct consequence of Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*), Definition 233 (*subalgebra*), Definition 226 (*algebra over a field*, $\mathcal{M}_{\mathbb{R}}$ is a vector space), and Lemma 236 (*closed under algebra operations is subalgebra*, $0_{\mathcal{M}_{\mathbb{R}}} = 0_{\mathbb{R}^X}$ is the zero function). □

10.2.2 Measurable numeric function to $\overline{\mathbb{R}}$

Definition 575 ***($\mathcal{M}$, set of measurable numeric functions).***
Let (X, Σ) be a measurable space. The set of functions $X \to \overline{\mathbb{R}}$ that are measurable for Σ and $\mathcal{B}(\overline{\mathbb{R}})$ is called the *set of measurable functions (over X)*; it is denoted $\mathcal{M}(X, \Sigma)$ (or simply $\mathcal{M}$).

Remark 576.
We use the convention of keeping the name of functions $X \to \mathbb{R}$ when they are "extended" as functions $X \to \overline{\mathbb{R}}$. This allows us to consider $\mathcal{M}_{\mathbb{R}}$ as a subset of $\mathcal{M}$ in the next lemma.

Lemma 577 ***($\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$).*** *Let (X, Σ) be a measurable space. Let $\mathbb{R}^X$ be the set of functions from X to $\mathbb{R}$. Then, we have $\mathcal{M}_{\mathbb{R}} = \mathcal{M} \cap \mathbb{R}^X$.*

Proof. $\boldsymbol{\mathcal{M}_{\mathbb{R}} \subset \mathcal{M} \cap \mathbb{R}^X}$. Let $f \in \mathcal{M}_{\mathbb{R}}$. Let $A \in \mathcal{B}(\overline{\mathbb{R}})$. Then, from Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*, $\overline{\mathbb{R}} = \mathbb{R} \uplus \{\pm\infty\}$), **compatibility of inverse image with union**, Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*, $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$), and Definition 522 (*measurable function*), we have $f \in \mathbb{R}^X$, and

$$f^{-1}(A) = f^{-1}(A \cap \mathbb{R}) \cup f^{-1}(A \cap \{\pm\infty\}) = f^{-1}(A \cap \mathbb{R}) \in \Sigma.$$

Hence, from Definition 522 (*measurable function*), and Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), $f \in \mathcal{M} \cap \mathbb{R}^X$.

$\boldsymbol{\mathcal{M} \cap \mathbb{R}^X \subset \mathcal{M}_{\mathbb{R}}}$. Let $f \in \mathcal{M} \cap \mathbb{R}^X$. Let $A \in \mathcal{B}(\mathbb{R})$. Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*, $A \cap \mathbb{R} = A \in \mathcal{B}(\mathbb{R})$), and Definition 522 (*measurable function*), we have $A \in \mathcal{B}(\overline{\mathbb{R}})$, and $f^{-1}(A) \in \Sigma$. Hence, from Definition 522 (*measurable function*), and Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*), we have $f \in \mathcal{M}_{\mathbb{R}}$.

Therefore, we have the equality. □

Lemma 578 ***(measurability of numeric function).*** *Let (X, Σ) be a measurable space. Let $f : X \to \overline{\mathbb{R}}$. Then, $f \in \mathcal{M}$ iff one of the following conditions is satisfied:*

$$\forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a) \in \Sigma, \tag{10.11}$$

$$\forall a \in \mathbb{R}, \quad \{f \leqslant a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a] \in \Sigma, \tag{10.12}$$

$$\forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty] \in \Sigma, \tag{10.13}$$

$$\forall a \in \mathbb{R}, \quad \{f \geqslant a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty] \in \Sigma. \tag{10.14}$$

Proof. **"Left" implies "right".** Direct consequence of Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Definition 522 (*measurable function*), Definition 517 (*Borel σ-algebra*), Lemma 518 (*some Borel subsets*), **open intervals of $\overline{\mathbb{R}}$ are open subsets**, and **closed intervals of $\overline{\mathbb{R}}$, are closed subsets**.

"Right" implies "left". Direct consequence of Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*), Lemma 528 (*equivalent definition of measurable function*), and Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*).

Therefore, we have the equivalence. □

Lemma 579 ***(inverse image is measurable).***
Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}$. Then, we have

$$\forall a \in \overline{\mathbb{R}}, \quad \{f = a\} \stackrel{\text{def.}}{=} f^{-1}(a) \in \Sigma. \tag{10.15}$$

Proof. Direct consequence of Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Definition 522 (*measurable function*), Lemma 518 (*some Borel subsets*, singletons are measurable). □

Lemma 580 ***($\mathcal{M}$ is closed under finite part).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}$. Let $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$. Then, $A \in \Sigma$ and (the finite part of f) $f\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}} (\subset \mathcal{M})$.*

Proof. From Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*), we have $\mathbb{R} = \{\pm\infty\}^c$. Let

$$B \stackrel{\text{def.}}{=} f^{-1}(\{\pm\infty\}) = f^{-1}(-\infty) \uplus f^{-1}(\infty).$$

Then, from **compatibility of inverse image with disjoint union**, Lemma 579 (*inverse image is measurable*), Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under union), we have $B \in \Sigma$. Hence, from **compatibility of inverse image with complement**, Definition 207 (*pseudopartition*), and Definition 474 (*σ-algebra*, closedness under complement), we have $A = B^c \in \Sigma$, and (A, B) form a pseudopartition of X.

From Definition 397 (*finite part*), let $\hat{f} \stackrel{\text{def.}}{=} f\mathbb{1}_A$ be the finite part of f. Therefore, from Lemma 398 (*finite part is finite*), **the definition of the indicator function**, Lemma 526 (*constant function is measurable*, $0 \in \mathcal{M}$), Lemma 539 (*measurability of function defined on a pseudopartition*, with $(X', \Sigma') \stackrel{\text{def.}}{=} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, and the pseudopartition $X = A \uplus B$), and Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*), we have $\hat{f} \in \mathcal{M} \cap \mathbb{R}^X = \mathcal{M}_{\mathbb{R}}$. □

Lemma 581 ***($\mathcal{M}$ is closed under addition when defined).***
Let (X, Σ) be a measurable space. Let $f, g \in \mathcal{M}$. Assume that f and g never take opposite infinite values at the same point. Then, $f + g$ is well-defined, and belongs to $\mathcal{M}$.

Proof. By hypothesis, the subsets $\{f = \infty\} \cap \{g = -\infty\}$ and $\{f = -\infty\} \cap \{g = \infty\}$ are empty. Thus, from Definition 207 (*pseudopartition*), the whole space reduces to the pseudopartition

$$X = X^c_{\pm\infty} \uplus X_\infty \uplus X_{-\infty}$$

(we recall that the notation $\uplus$ is for disjoint union) where

$$\begin{aligned} X^{\mathrm{c}}_{\pm\infty} &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}),\\ X_{\infty} &\stackrel{\text{def.}}{=} \{f=\infty\} \cup \{g=\infty\},\\ X_{-\infty} &\stackrel{\text{def.}}{=} \{f=-\infty\} \cup \{g=-\infty\}. \end{aligned}$$

Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Definition 522 (*measurable function*, for f and g), Definition 517 (*Borel σ-algebra*, $\mathbb{R}$ open in $\overline{\mathbb{R}}$ belongs to $\mathcal{B}(\overline{\mathbb{R}})$), Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable union and intersection), and Lemma 579 (*inverse image is measurable*, with $a \stackrel{\text{def.}}{=} \pm\infty$), $X^{\mathrm{c}}_{\pm\infty}$, X_{∞} and $X_{-\infty}$ belong to Σ.

Let $\tilde{f}$ and $\tilde{g}$ be the finite parts of f and g. Then, from Definition 397 (*finite part*), and Definition 282 (*addition in $\overline{\mathbb{R}}$*), the sum $f+g$ can be defined on the pseudopartition by

$$f+g \stackrel{\text{def.}}{=} \begin{cases} \tilde{f}+\tilde{g} & \text{on } X^{\mathrm{c}}_{\pm\infty},\\ \infty & \text{on } X_{\infty},\\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 580 (*$\mathcal{M}$ is closed under finite part*, with f and g), Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*, sum), Lemma 526 (*constant function is measurable*), and Lemma 539 (*measurability of function defined on a pseudopartition*), we have $f+g \in \mathcal{M}$. ☐

Lemma 582 ***($\mathcal{M}$ is closed under finite sum when defined).***
Let (X,Σ) be a measurable space. Let I be a finite set. Let $(f_i)_{i\in I} \in \mathcal{M}$. Assume that the f_i's never take opposite infinite values at the same point, i.e. for all $i,j \in I$, $i \neq j$ implies $\{f_i=\infty\} \cap \{f_j=-\infty\} = \varnothing$. Then, $\sum_{i\in I} f_i$ is well-defined, and belongs to $\mathcal{M}$.

Proof. Direct consequence of Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*), and induction on the cardinality of I. ☐

Lemma 583 ***($\mathcal{M}$ is closed under multiplication).***
Let (X,Σ) be a measurable space. Let $f,g \in \mathcal{M}$. Then, fg is well-defined, and belongs to $\mathcal{M}$.

Proof. From Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), and Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*), the whole space can be pseudopartitioned into $X = X^{\mathrm{c}}_{\pm\infty} \uplus X_0 \uplus X_{\infty} \uplus X_{-\infty}$ (the notation $\uplus$ is for disjoint union) where

$$\begin{aligned} X^{\mathrm{c}}_{\pm\infty} &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}),\\ X_0 &\stackrel{\text{def.}}{=} (\{|f|=\infty\} \cap \{g=0\}) \cup (\{f=0\} \cap \{|g|=\infty\}),\\ X_{\infty} &\stackrel{\text{def.}}{=} (\{f=\infty\} \cap \{g>0\}) \cup (\{f=-\infty\} \cap \{g<0\})\\ &\quad \cup (\{f>0\} \cap \{g=\infty\}) \cup (\{f<0\} \cap \{g=-\infty\}),\\ X_{-\infty} &\stackrel{\text{def.}}{=} (\{f=-\infty\} \cap \{g>0\}) \cup (\{f=\infty\} \cap \{g<0\})\\ &\quad \cup (\{f>0\} \cap \{g=-\infty\}) \cup (\{f<0\} \cap \{g=\infty\}). \end{aligned}$$

Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Definition 522 (*measurable function*, for f and g), Definition 517 (*Borel σ-algebra*, $\mathbb{R}$ open in $\overline{\mathbb{R}}$ belongs to $\mathcal{B}(\overline{\mathbb{R}})$), Lemma 579 (*inverse image is measurable*, with $a \stackrel{\text{def.}}{=} \pm\infty, 0$) Lemma 578 (*measurability of numeric function*, with $a \stackrel{\text{def.}}{=} 0$), and Lemma 475 (*equivalent definition of σ-algebra*)closedness under countable union and intersection, we have $X^{\mathrm{c}}_{\pm\infty}, X_0, X_{\infty}, X_{-\infty} \in \Sigma$.

Let $\tilde{f}$ and $\tilde{g}$ be the finite parts of f and g. Then, from Definition 397 (*finite part*), Definition 288 (*multiplication in* $\overline{\mathbb{R}}$), and Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*), the product fg can be defined on the pseudopartition by

$$fg \stackrel{\text{def.}}{=} \begin{cases} \tilde{f}\tilde{g} & \text{on } X_{\pm\infty}^c, \\ 0 & \text{on } X_0, \\ \infty & \text{on } X_\infty, \\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 580 ($\mathcal{M}$ *is closed under finite part*, with f and g), Lemma 572 ($\mathcal{M}_\mathbb{R}$ *is algebra*, product), Lemma 526 (*constant function is measurable*), and Lemma 539 (*measurability of function defined on a pseudopartition*), we have $fg \in \mathcal{M}$. □

Lemma 584 ***($\mathcal{M}$ is closed under finite product).***
Let (X, Σ) be a measurable space. Let I be a finite set. Let $(f_i)_{i\in I} \in \mathcal{M}$. Then, $\prod_{i\in I} f_i \in \mathcal{M}$.

Proof. Direct consequence of Lemma 583 ($\mathcal{M}$ *is closed under multiplication*), and induction on the cardinality of I. □

Lemma 585 ***($\mathcal{M}$ is closed under scalar multiplication).***
Let (X, Σ) be a measurable space. Let $a \in \overline{\mathbb{R}}$. Let $f \in \mathcal{M}$. Then, we have $af \in \mathcal{M}$.

Proof. Direct consequence of Lemma 583 ($\mathcal{M}$ *is closed under multiplication*), and Lemma 526 (*constant function is measurable*). □

Lemma 586 ***($\mathcal{M}$ is closed under infimum).***
Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(f_i)_{i\in I} \in \mathcal{M}$. Then, we have $\inf_{i\in I} f_i \in \mathcal{M}$.

Proof. Let $a \in \mathbb{R}$. Then, from Definition 9 (*infimum*), Lemma 578 (*measurability of numeric function*, with f_i), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have

$$\left\{\inf_{i\in I} f_i \geqslant a\right\} = \bigcap_{i\in I}\{f_i \geqslant a\} \in \Sigma.$$

Therefore, from Lemma 578 (*measurability of numeric function*, with $\inf_{i\in I} f_i$), $\inf_{i\in I} f_i$ belongs to $\mathcal{M}$. □

Lemma 587 ***($\mathcal{M}$ is closed under supremum).***
Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(f_i)_{i\in I} \in \mathcal{M}$. Then, we have $\sup_{i\in I} f_i \in \mathcal{M}$.

Proof. Let $a \in \mathbb{R}$. Then, from Definition 2 (*supremum*), Lemma 578 (*measurability of numeric function*, with all the f_i's), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection),

$$\left\{\sup_{i\in I} f_i \leqslant a\right\} = \bigcap_{i\in I}\{f_i \leqslant a\} \in \Sigma.$$

Therefore, from Lemma 578 (*measurability of numeric function*, with $\sup_{i\in I} f_i$), $\sup_{i\in I} f_i$ belongs to $\mathcal{M}$. □

Lemma 588 ***($\mathcal{M}$ is closed under limit inferior).***
Let (X, Σ) be a measurable space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}$. Then, we have $\liminf_{n\to\infty} f_n \in \mathcal{M}$.

Proof. Direct consequence of Lemma 586 ($\mathcal{M}$ *is closed under infimum*, $F_n^- \stackrel{\text{def.}}{=} \inf_{p\in\mathbb{N}} f_{n+p} \in \mathcal{M}$), Lemma 587 ($\mathcal{M}$ *is closed under supremum*, $\underline{f} \stackrel{\text{def.}}{=} \sup_{n\in\mathbb{N}} F_n^- \in \mathcal{M}$), and Lemma 378 (*limit inferior*, $\liminf_{n\to\infty} f_n = \underline{f}$). □

Lemma 589 *($\mathcal{M}$ is closed under limit superior).*
Let (X, Σ) be a measurable space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}$. Then, we have $\limsup_{n\to\infty} f_n \in \mathcal{M}$.

Proof. Direct consequence of Lemma 587 (*$\mathcal{M}$ is closed under supremum*, $F_n^+ \stackrel{\text{def.}}{=} \sup_{p\in\mathbb{N}} f_{n+p}$ belongs to $\mathcal{M}$), Lemma 586 (*$\mathcal{M}$ is closed under infimum*, $\overline{f} \stackrel{\text{def.}}{=} \inf_{n\in\mathbb{N}} F_n^+ \in \mathcal{M}$), and Lemma 383 (*limit superior*, $\limsup_{n\to\infty} f_n = \overline{f}$). □

Lemma 590 *($\mathcal{M}$ is closed under limit when pointwise convergent).*
Let (X, Σ) be a measurable space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}$. Assume that $(f_n(x))_{n\in\mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}$. Then, we have $\lim_{n\to\infty} f_n \in \mathcal{M}$.

Proof. Direct consequence of Lemma 396 (*limit inferior, limit superior and pointwise convergence*), Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*), or Lemma 589 (*$\mathcal{M}$ is closed under limit superior*). □

Lemma 591 *(measurability and masking).*
Let (X, Σ) be a measurable space. Let $f : X \to \overline{\mathbb{R}}$. Then, $f \in \mathcal{M}$ iff for all $A \in \Sigma$, $f\mathbb{1}_A \in \mathcal{M}$.

Proof. **"Left" implies "right".** Direct consequence of Lemma 569 (*measurability of indicator function*), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_\mathbb{R}$*, $\mathcal{M}_\mathbb{R} \subset \mathcal{M}$), and Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

"Right" implies "left". Direct consequence of Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $X \in \Sigma$), **the definition of the indicator function ($\mathbb{1}_X \equiv 1$)**, and Definition 288 (*multiplication in $\overline{\mathbb{R}}$*, 1 is unity).

Therefore, we have the equivalence. □

Lemma 592 *(measurability of restriction).*
Let (X, Σ) be a measurable space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, we have $f_{|A} \in \mathcal{M}(A, \Sigma \overline{\cap} A)$ iff $\hat{f}\mathbb{1}_A \in \mathcal{M}(X, \Sigma)$.

Proof. Let $B' \in \mathcal{B}(\overline{\mathbb{R}})$. **Case $0 \not\in B'$.** Then, from **the definition of the indicator function**, we have $(\hat{f}\mathbb{1}_A)^{-1}(B') = f^{-1}(B') \cap A$. **Case $0 \in B'$.** Then, from **compatibility of inverse image with disjoint union**, **the definition of the indicator function**, and **associativity of disjoint union**, we have

$$\begin{aligned}
(\hat{f}\mathbb{1}_A)^{-1}(B') &= (\hat{f}\mathbb{1}_A)^{-1}(B' \setminus \{0\}) \uplus (\hat{f}\mathbb{1}_A)^{-1}(0) \\
&= \left(f^{-1}(B' \setminus \{0\}) \cap A\right) \uplus \left(\left(f^{-1}(0) \cap A\right) \uplus A^c\right) \\
&= \left(f^{-1}(B') \cap A\right) \uplus A^c.
\end{aligned}$$

Moreover, from **the definition of restriction of function**, we have $f_{|A}^{-1}(B') = f^{-1}(B') \cap A$. Hence, we have $(\hat{f}\mathbb{1}_A)^{-1}(B') = f_{|A}^{-1}(B') \uplus C$ with $C \in \{\varnothing, A^c\}$.

Let $B \subset A$. Assume first that $B \in \Sigma$. Then, from Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under complement and countable union), we have $B \uplus A^c \in \Sigma$. Conversely, assume now that $B \uplus A^c \in \Sigma$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have $(B \uplus A^c) \cap A = B \in \Sigma$. Hence, we have the equivalence $B \in \Sigma$ iff $B \uplus A^c \in \Sigma$.

Therefore, from Definition 522 (*measurable function*), and Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), we have the equivalence $f_{|A} \in \mathcal{M}(A, \Sigma \overline{\cap} A)$ iff $\hat{f}\mathbb{1}_A \in \mathcal{M}(X, \Sigma)$. □

10.2.3 Nonnegative measurable numeric function

Definition 593 ***($\mathcal{M}_+$, subset of nonnegative measurable numeric functions).***
Let (X, Σ) be a measurable space. The *subset of nonnegative measurable functions (over X)* is denoted $\mathcal{M}_+(X, \Sigma)$ (or simply $\mathcal{M}_+$); it is defined by $\mathcal{M}_+(X, \Sigma) \stackrel{\text{def.}}{=} \{f \in \mathcal{M} \mid f(X) \subset \overline{\mathbb{R}}_+\}$.

Lemma 594 ***(measurability of nonnegative and nonpositive parts).***
Let (X, Σ) be a measurable space. Let $f : X \to \overline{\mathbb{R}}$. Then, we have $f \in \mathcal{M}$ iff $f^+, f^- \in \mathcal{M}_+$.

Proof. **"Left" implies "right".** Direct consequence of Definition 399 (*nonnegative and nonpositive parts*), Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*, $-f \in \mathcal{M}$), Lemma 526 (*constant function is measurable*, $0 \in \mathcal{M}$), Lemma 587 (*$\mathcal{M}$ is closed under supremum*, maximum is supremum), Lemma 401 (*nonnegative and nonpositive parts are nonnegative*), and Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

"Right" implies "left". Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*, $\mathcal{M}_+ \subset \mathcal{M}$), Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*, $-f^- \in \mathcal{M}$), Lemma 403 (*decomposition into nonnegative and nonpositive parts*, $f^+ - f^-$ is well-defined), and Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*).

Therefore, we have the equivalence. □

Lemma 595 ***($\mathcal{M}_+$ is closed under finite part).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}_+$. Let $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$. Then, $A \in \Sigma$ and (the finite part of f) $f \mathbb{1}_A \in \mathcal{M}_\mathbb{R} \cap \mathcal{M}_+$.*

Proof. Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 580 (*$\mathcal{M}$ is closed under finite part*), **nonnegativeness of the indicator function**, and **closedness of multiplication in** $\mathbb{R}_+$. □

Lemma 596 ***($\mathcal{M}$ is closed under absolute value).***
Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}$. Then, we have $|f| \in \mathcal{M}_+ \subset \mathcal{M}$.
Moreover, if $f \in \mathcal{M}_\mathbb{R}$, then we have $|f| \in \mathcal{M}_\mathbb{R} \cap \mathcal{M}_+$.

Proof. Direct consequence of Lemma 317 (*absolute value in $\overline{\mathbb{R}}$ is continuous*), Lemma 529 (*continuous is measurable*), Lemma 530 (*compatibility of measurability with composition*), Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*, $|f| \in \mathcal{M}$), Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*), Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*, $|f| \in \mathcal{M}_+$), **closedness of absolute value in** $\mathbb{R}$, and Definition 567 (*$\mathcal{M}_\mathbb{R}$, vector space of measurable numeric functions to* $\mathbb{R}$, $|f| \in \mathcal{M}_\mathbb{R}$). □

Lemma 597 ***($\mathcal{M}_+$ is closed under addition).*** *Let (X, Σ) be a measurable space. Let $f, g \in \mathcal{M}_+$. Then, $f + g$ is well-defined, and belongs to $\mathcal{M}_+$.*

Proof. Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*, f and g cannot take opposite infinite values), and Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*). □

Lemma 598 ***($\mathcal{M}_+$ is closed under multiplication).***
Let (X, Σ) be a measurable space. Let $f, g \in \mathcal{M}_+$. Then, we have $fg \in \mathcal{M}_+$.

Proof. Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 583 (*$\mathcal{M}$ is closed under multiplication*), and Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*). □

Lemma 599 ***($\mathcal{M}_+$ is closed under nonnegative scalar multiplication).***
Let (X, Σ) be a measurable space. Let $a \in \overline{\mathbb{R}}_+$. Let $f \in \mathcal{M}_+$. Then, we have $af \in \mathcal{M}_+$.

Proof. Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*), and Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*). □

Lemma 600 ***($\mathcal{M}_+$ is closed under infimum).***
Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(f_i)_{i\in I} \in \mathcal{M}_+$. Then, we have $\inf_{i\in I} f_i \in \mathcal{M}_+$.

Proof. Direct consequence of Lemma 586 (*$\mathcal{M}$ is closed under infimum*), and Lemma 376 (*infimum of bounded sequence is bounded*, with $a \stackrel{\text{def.}}{=} 0$). □

Lemma 601 ***($\mathcal{M}_+$ is closed under supremum).***
Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(f_i)_{i\in I} \in \mathcal{M}_+$. Then, we have $\sup_{i\in I} f_i \in \mathcal{M}_+$.

Proof. Direct consequence of Lemma 587 (*$\mathcal{M}$ is closed under supremum*), and Lemma 377 (*supremum of bounded sequence is bounded*, with $a \stackrel{\text{def.}}{=} 0$). □

Lemma 602 ***($\mathcal{M}_+$ is closed under limit when pointwise convergent).***
Let (X, Σ) be a measurable space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Assume that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}_+$. Then, we have $\lim_{n\to\infty} f_n \in \mathcal{M}_+$.

Proof. Direct consequence of Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*), and **completeness of $\overline{\mathbb{R}}_+$**. □

Lemma 603 ***($\mathcal{M}_+$ is closed under countable sum).*** *Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(f_i)_{i\in I} \in \mathcal{M}_+$. Then, $\sum_{i\in I} f_i$ is well-defined, and belongs to $\mathcal{M}_+$.*

Proof. Direct consequence of Lemma 322 (*series are convergent in $\overline{\mathbb{R}}_+$*), Lemma 597 (*$\mathcal{M}_+$ is closed under addition*), induction on $n \in \mathbb{N}$ (with $g_n \stackrel{\text{def.}}{=} \sum_{i\in I\cap[0..n]} f_i \in \mathcal{M}_+$, with the convention that a sum indexed by $\varnothing$ is 0), and Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*, with $f_n \stackrel{\text{def.}}{=} g_n$). □

10.2.4 Tensor product of measurable numeric functions

Definition 604 ***(tensor product of numeric functions).***
Let $m \in [2..\infty)$. For all $i \in [1..m]$, let X_i be a set, and $f_i : X_i \to \overline{\mathbb{R}}$. Let $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$. The *tensor product of* $(f_i)_{i\in[1..m]}$ is the function $\bigotimes_{i\in[1..m]} f_i : X \to \overline{\mathbb{R}}$ defined by

$$\forall x \stackrel{\text{def.}}{=} (x_i)_{i\in[1..m]} \in X, \quad \left(\bigotimes_{i\in[1..m]} f_i\right)(x) \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} f_i(x_i). \tag{10.16}$$

Lemma 605 ***(measurability of tensor product of numeric functions).***
Let $m \in [2..\infty)$. For all $i \in [1..m]$, let (X_i, Σ_i) be a measurable space, and $f_i \in \mathcal{M}(X_i, \Sigma_i)$. Then, we have $\bigotimes_{i\in[1..m]} f_i \in \mathcal{M}\left(\prod_{i\in[1..m]} X_i, \bigotimes_{i\in[1..m]} \Sigma_i\right)$.

Proof. Let $X \stackrel{\text{def.}}{=} \prod_{i\in[1..m]} X_i$ and $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i\in[1..m]} \Sigma_i$. Let $i \in [1..m]$. Let π_i be the canonical projection from X onto X_i. Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Lemma 544 (*canonical projection is measurable*), and Lemma 530 (*compatibility of measurability* *with composition*), $f_i \circ \pi_i$ belongs to $\mathcal{M}(X,\Sigma)$. Therefore, from Definition 604 (*tensor product of numeric functions*), and Lemma 584 (*$\mathcal{M}$ is closed under finite product*, with $I \stackrel{\text{def.}}{=} [1..m]$), we have

$$\bigotimes_{i\in[1..m]} f_i = \prod_{i\in[1..m]} (f_i \circ \pi_i) \in \mathcal{M}(X,\Sigma).$$

$\square$

Chapter 11

Measure space

Contents

11.1 Measure

Remark 606. We recall that $\uplus$ denotes disjoint union.

Definition 607 ***(additivity over measurable space).*** Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is said *additive* iff for all $n \in \mathbb{N}$, for all $(A_i)_{i\in[0..n]} \in \Sigma$,

$$(\forall p, q \in [0..n],\ p \neq q \Rightarrow A_p \cap A_q = \varnothing) \implies \mu\left(\biguplus_{i\in[0..n]} A_i\right) = \sum_{i\in[0..n]} \mu(A_i). \tag{11.1}$$

Definition 608 ***(σ-additivity over measurable space).*** Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is said *σ-additive* iff for all $I \subset \mathbb{N}$, for all $(A_i)_{i\in I} \in \Sigma$,

$$(\forall p, q \in I,\ p \neq q \Rightarrow A_p \cap A_q = \varnothing) \implies \mu\left(\biguplus_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i). \tag{11.2}$$

Remark 609. Note that from Definitions 516 and 474 (closedness under countable union), both previous definitions are well-defined.

Lemma 610 ***(σ-additivity implies additivity).*** *Let (X, Σ) be a measurable space. Let $\mu : \Sigma \to \overline{\mathbb{R}}$. Assume that μ is σ-additive. Then, μ is additive.*

Proof. Direct consequence of Definition 608 (*σ-additivity over measurable space*, with $I \overset{\text{def.}}{=} [0..n]$), and Definition 607 (*additivity over measurable space*). □

Definition 611 ***(measure).***
Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is called *measure on* (X, Σ) iff it is nonnegative, $\mu(\varnothing) = 0$, and it is σ-additive. If so, (X, Σ, μ) is called *measure space*.

Remark 612. The previous definition is actually that of *nonnegative* measure, but as we do not consider "signed" measures in this document, we omit the qualifier "nonnegative".

Lemma 613 ***(measure over countable pseudopartition).***
Let (X, Σ, μ) be a measure space. Let $I \subset \mathbb{N}$. Let $A, (B_i)_{i \in I} \in \Sigma$. Assume that $X = \biguplus_{i \in I} B_i$. Then, for all $i \in I$, $A \cap B_i = B_i \cap A \in \Sigma$, and we have

$$\mu(A) = \sum_{i \in I} \mu(A \cap B_i) = \sum_{i \in I} \mu(B_i \cap A). \tag{11.3}$$

Proof. Direct consequence of Lemma 209 (*compatibility of pseudopartition with intersection*), Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection with card(I) equals 2), Definition 611 (*measure*, σ-additive), Definition 608 (*σ-additivity over measurable space*), and **commutativity of intersection**. □

Lemma 614 ***(measure is monotone).***
Let (X, Σ, μ) be a measure space. Then, μ is nondecreasing over Σ:

$$\forall A, B \in \Sigma, \quad A \subset B \quad \Longrightarrow \quad \mu(A) \leqslant \mu(B). \tag{11.4}$$

Moreover, if $\mu(A)$ is finite, then we have $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proof. Let $A, B \in \Sigma$. Assume that $A \subset B$. Then, from **the definition of set difference**, and Lemma 478 (*σ-algebra is closed under set difference*), we have $A \cap (B \setminus A) = \varnothing$, $B = A \uplus (B \setminus A)$ and $B \setminus A \in \Sigma$. Therefore, from Definition 611 (*measure*, nonnegativeness), and Definition 608 (*σ-additivity over measurable space*) we have $\mu(B) = \mu(A) + \mu(B \setminus A) \geqslant \mu(A)$.

Assume now that $\mu(A) < \infty$. Then, from Definition 282 (*addition in $\overline{\mathbb{R}}$*, rule 5 applies), we have $\mu(B \setminus A) = \mu(B) - \mu(A)$. □

Lemma 615 ***(measure satisfies the finite Boole inequality).***
Let (X, Σ, μ) be a measure space. Then, μ satisfies the finite Boole inequality:

$$\forall n \in \mathbb{N},\ \forall (A_i)_{i \in [0..n]} \in \Sigma, \quad \mu\left(\bigcup_{i \in [0..n]} A_i\right) \leqslant \sum_{i \in [0..n]} \mu(A_i). \tag{11.5}$$

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the property $\forall (A_i)_{i \in [0..n]} \in \Sigma,\ \mu\left(\bigcup_{i \in [0..n]} A_i\right) \leqslant \sum_{i \in [0..n]} \mu(A_i)$.

Induction: $P(0)$. Trivial.

Induction: $P(n)$ implies $P(n+1)$. Let $n \in \mathbb{N}$. Assume that $P(n)$ holds.
For all $i \in [0..(n+1)]$, let $A_i \in \Sigma$. From Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under countable union), let $B \overset{\text{def.}}{=} \bigcup_{i \in [0..n]} A_i \in \Sigma$. Then, from **the definition of set difference**, **associativity of union**, and Lemma 478 (*σ-algebra is closed under set difference*), we have

$$B \cap (A_{n+1} \setminus B) = \varnothing, \quad \bigcup_{i \in [0..(n+1)]} A_i = B \uplus (A_{n+1} \setminus B), \quad \text{and} \quad A_{n+1} \setminus B \in \Sigma.$$

Thus, from Definition 611 (*measure*, σ-additive), Definition 608 (*σ-additivity over measurable space*, with card$(I) = 2$), $P(n)$, Lemma 614 (*measure is monotone*, with $A_{n+1} \setminus B \subset A_{n+1}$), and **monotonicity of addition**, we have

$$\mu\left(\bigcup_{i \in [0..(n+1)]} A_i\right) = \mu(B \uplus (A_{n+1} \setminus B)) = \mu(B) + \mu(A_{n+1} \setminus B) \leqslant \sum_{i \in [0..(n+1)]} \mu(A_i).$$

Hence, $P(n+1)$ holds.

Therefore, by induction, we have $P(n)$ for all $n \in \mathbb{N}$. □

Definition 616 ***(continuity from below).***
Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is said *continuous from below* iff

$$\forall (A_n)_{n\in\mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N},\ A_n \subset A_{n+1}) \quad \Longrightarrow \quad \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n). \tag{11.6}$$

Lemma 617 ***(measure is continuous from below).***
Let (X, Σ, μ) be a measure space. Then, μ is continuous from below. Moreover, we have

$$\forall (A_n)_{n\in\mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N},\ A_n \subset A_{n+1}) \quad \Longrightarrow \quad \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\in\mathbb{N}} \mu(A_n). \tag{11.7}$$

Proof. Let $(A_n)_{n\in\mathbb{N}} \in \Sigma$. Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ is nondecreasing.

Let $B_0 \stackrel{\text{def.}}{=} A_0 \in \Sigma$. for all $n \in \mathbb{N}$, let $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i\in[0..n]} B_i$. Then, from Lemma 480 (*partition of countable union in σ-algebra*), and **partial union law for nondecreasing sequence**, the sequence $(B_n)_{n\in\mathbb{N}}$ is pairwise disjoint, and we have

$$\forall n \in \mathbb{N}, \quad B_n \in \Sigma \quad \wedge \quad \biguplus_{i\in[0..n]} B_i = \bigcup_{i\in[0..n]} A_i = A_n \in \Sigma,$$
$$\bigcup_{n\in\mathbb{N}} A_n = \biguplus_{n\in\mathbb{N}} B_n \in \Sigma.$$

Hence, from Definition 611 (*measure*, σ-additivity), Definition 608 (*σ-additivity over measurable space*), and **the definition of the sum of a sequence of positive numbers**, we have

$$\begin{aligned}\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\biguplus_{n\in\mathbb{N}} B_n\right) &= \sum_{n\in\mathbb{N}} \mu(B_n)\\ &= \lim_{n\to\infty} \sum_{i\in[0..n]} \mu(B_i) = \lim_{n\to\infty} \mu\left(\biguplus_{i\in[0..n]} B_n\right) = \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Therefore, from Definition 616 (*continuity from below*), μ is continuous from below.

Moreover, Equation (11.7) is a direct consequence of Lemma 614 (*measure is monotone*), and **properties of nondecreasing sequences in $\overline{\mathbb{R}}_+$**. □

Definition 618 ***(continuity from above).***
Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is said *continuous from above* iff

$$\forall (A_n)_{n\in\mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N},\ A_n \supset A_{n+1} \quad \wedge \quad \exists n_0 \in \mathbb{N},\ \mu(A_{n_0}) < \infty) \tag{11.8}$$

$$\Longrightarrow \quad \mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n) < \infty. \tag{11.9}$$

Lemma 619 ***(measure is continuous from above).***
Let (X, Σ, μ) be a measure space. Then, μ is continuous from above. Moreover, we have

$$\forall (A_n)_{n\in\mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N},\ A_n \supset A_{n+1} \quad \wedge \quad \exists n_0 \in \mathbb{N},\ \mu(A_{n_0}) < \infty) \tag{11.10}$$

$$\Longrightarrow \quad \mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \inf_{n\in\mathbb{N}} \mu(A_n) < \infty. \tag{11.11}$$

Proof. Let $(A_n)_{n\in\mathbb{N}} \in \Sigma$. Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ is nonincreasing. Then, from **properties of the intersection and of the limit**, we have

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\geqslant n_0} A_n \quad \text{and} \quad \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_{n_0+n}).$$

For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} A_{n_0} \setminus A_{n_0+n}$ (thus $B_0 = \varnothing$). Then, from **the definition and properties of set difference**, Lemma 478 (*σ-algebra is closed under set difference*), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), $(B_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of measurable subsets such that $\bigcup_{n\in\mathbb{N}} B_n = A_{n_0} \setminus \bigcap_{n\geqslant n_0} A_n \in \Sigma$. Thus, from Lemma 617 (*measure is continuous from below*), Definition 616 (*continuity from below*), and Lemma 614 (*measure is monotone*, with $\mu(A_{n_0}) < \infty$), we have

$$\mu(A_{n_0}) - \mu\left(\bigcap_{n\geqslant n_0} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \left(\mu(A_{n_0}) - \mu(A_{n_0+n})\right).$$

Hence, from **linearity of the limit**, **additive group properties of $\mathbb{R}$**, and Lemma 614 (*measure is monotone*), we have

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcap_{n\geqslant n_0} A_n\right) = \lim_{n\to\infty} \mu(A_{n_0+n}) = \lim_{n\to\infty} \mu(A_n) \leqslant \mu(A_{n_0}) < \infty.$$

Therefore, from Definition 618 (*continuity from above*), μ is continuous from above.

Moreover, Equation (11.10) is a direct consequence of Lemma 614 (*measure is monotone*), and **properties of nonincreasing sequences in $\overline{\mathbb{R}}_+$**. □

Lemma 620 ***(measure satisfies the Boole inequality).***
Let (X, Σ, μ) be a measure space. Then, μ satisfies the Boole inequality:

$$\forall (A_n)_{n\in\mathbb{N}} \in \Sigma, \quad \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_{n\in\mathbb{N}} \mu(A_n). \tag{11.12}$$

Proof. Let $(A_n)_{n\in\mathbb{N}} \in \Sigma$. For all $n \in \mathbb{N}$, from Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under countable union), let $B_n \stackrel{\text{def.}}{=} \bigcup_{i\in[0..n]} A_i \in \Sigma$. Then, from **properties of union**, and Definition 474 (*σ-algebra*, closedness under countable union), the sequence $(B_n)_{n\in\mathbb{N}}$ is nondecreasing and

$$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n \in \Sigma.$$

Thus, from Lemma 617 (*measure is continuous from below*), we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sup_{n\in\mathbb{N}} \mu(B_n).$$

Let $n \in \mathbb{N}$. Then, from Lemma 615 (*measure satisfies the finite Boole inequality*), and Definition 611 (*measure*, nonnegativeness), we have

$$\mu(B_n) = \mu\left(\bigcup_{i\in[0..n]} A_i\right) \leqslant \sum_{i\in[0..n]} \mu(A_i) \leqslant \sum_{n\in\mathbb{N}} \mu(A_n).$$

Therefore, from Definition 2 (*supremum*, least upper bound), we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_{n\in\mathbb{N}} \mu(A_n).$$

□

Lemma 621 ***(equivalent definition of measure).***
Let (X,Σ) be a measurable space. Let $\mu : \Sigma \to \overline{\mathbb{R}}$. Assume that μ is nonnegative, and that $\mu(\varnothing) = 0$. Then, μ is a measure on (X,Σ) iff it is additive and continuous from below.

Proof. **"Left" implies "right".** Direct consequence of Definition 611 (*measure*), Lemma 610 (*σ-additivity implies additivity*), and Lemma 617 (*measure is continuous from below*).

"Right" implies "left". Assume that μ is additive and continuous from below.
Let $I \subset \mathbb{N}$. Let $(A_i)_{i\in I} \in \Sigma$. Assume that for all $p,q \in I$, $p \neq q \Rightarrow A_p \cap A_q = \varnothing$. For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} \biguplus_{i\in I, i\in[0..n]} A_i$. Then, from Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under countable union), $(B_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of measurable subsets such that $\bigcup_{n\in\mathbb{N}} B_n = \biguplus_{i\in I} A_i \in \Sigma$. Thus, from Definition 616 (*continuity from below*), and Definition 607 (*additivity over measurable space*),

$$\begin{aligned}\mu\left(\biguplus_{i\in I} A_i\right) = \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \lim_{n\to\infty} \mu(B_n)\\ &= \lim_{n\to\infty} \mu\left(\biguplus_{i\in I\cap[0..n]} A_i\right) = \lim_{n\to\infty} \sum_{i\in I\cap[0..n]} \mu(A_i) = \sum_{i\in I} \mu(A_i).\end{aligned}$$

Hence, from Definition 608 (*σ-additivity over measurable space*), and Definition 611 (*measure*), μ is a measure on (X,Σ).

Therefore, we have the equivalence. □

Definition 622 ***(finite measure).*** Let (X,Σ,μ) be a measure space.
The measure μ is said *finite* iff $\mu(X) < \infty$. If so, the measure space (X,Σ,μ) is also said *finite.*

Lemma 623 ***(finite measure is bounded).***
Let (X,Σ,μ) be a finite measure space. Then, μ is bounded.

Proof. Direct consequence of Lemma 475 (*equivalent definition of σ-algebra*, contains full set), and Lemma 614 (*measure is monotone*, $\mu(A) \leqslant \mu(X) < \infty$). □

Definition 624 ***(σ-finite measure).***
Let (X,Σ,μ) be a measure space. The measure μ is said *σ-finite* iff

(11.13) $$\exists (A_n)_{n\in\mathbb{N}} \in \Sigma,\quad (\forall n \in \mathbb{N},\quad \mu(A_n) < \infty) \quad\wedge\quad X = \bigcup_{n\in\mathbb{N}} A_n.$$

If so, the measure space (X,Σ,μ) is also said *σ-finite.*

Lemma 625 ***(equivalent definition of σ-finite measure).***
Let (X,Σ,μ) be a measure space. Then, the measure μ is σ-finite iff

(11.14) $$\exists (B_n)_{n\in\mathbb{N}} \in \Sigma,\quad (\forall n \in \mathbb{N},\quad B_n \subset B_{n+1} \quad\wedge\quad \mu(B_n) < \infty) \quad\wedge\quad X = \bigcup_{n\in\mathbb{N}} B_n.$$

Proof. **"Left" implies "right".** From Definition 624 (*σ-finite measure*), there exists $(A_n)_{n\in\mathbb{N}} \in \Sigma$ such that, for all $n \in \mathbb{N}$, $\mu(A_n) < \infty$, and $X = \bigcup_{n\in\mathbb{N}} A_n$. For all $n \in \mathbb{N}$, from Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under countable union), let $B_n \overset{\text{def.}}{=} \bigcup_{p\in[0..n]} A_p \in \Sigma$.

Let $n \in \mathbb{N}$. Then, from **associativity of union**, we have $B_{n+1} = B_n \cup A_{n+1}$. Thus, we have $B_n \subset B_{n+1}$, and $\bigcup_{p\in[0..n]} B_p = B_n = \bigcup_{p\in[0..n]} A_n$. Hence, from Lemma 615 (*measure satisfies the finite Boole inequality*), **closedness of addition in $\mathbb{R}_+$**, and **the definition of countable union**, we have, for all $n \in \mathbb{N}$, $\mu(B_n) \leqslant \sum_{p\in[0..n]} \mu(A_p) < \infty$, and $X = \bigcup_{n\in\mathbb{N}} B_n$.

"Right" implies "left". Trivial.

Therefore, we have the equivalence. □

Definition 626 ***(diffuse measure).***
Let (X, Σ, μ) be a measure space. Assume that Σ contains all singletons of X. The measure μ is said *diffuse* iff for all $x \in X$, $\mu(\{x\}) = 0$. If so, the measure space (X, Σ, μ) is also said *diffuse*.

Lemma 627 ***(finite measure is σ-finite).***
Let (X, Σ, μ) be a finite measure space. Then, μ is σ-finite.

Proof. Direct consequence of Definition 624 (*σ-finite measure*, with $A_n = X$), and Definition 622 (*finite measure*). □

Lemma 628 ***(trace measure).***
Let (X, Σ, μ) be a measure space. Let $Y \in \Sigma$. Then, $\mu_{|\Sigma \bar{\cap} Y}$ is a measure on $(Y, \Sigma \bar{\cap} Y)$.

The measure $\mu_Y \overset{\text{def.}}{=} \mu_{|\Sigma \bar{\cap} Y}$ is called trace measure on Y. *The measure space $(Y, \Sigma \bar{\cap} Y, \mu_Y)$ is called* trace measure space on Y.

Proof. Direct consequence of Lemma 532 (*trace σ-algebra*), Lemma 533 (*measurability of measurable subspace*, $\Sigma \bar{\cap} Y \subset \Sigma$), and Definition 611 (*measure*). □

Lemma 629 ***(restricted measure).***
Let (X, Σ, μ) be a measure space. Let $Y \in \Sigma$. Then, the function μ'_Y defined on Σ by

$$\forall A \in \Sigma, \quad \mu'_Y(A) \overset{\text{def.}}{=} \mu(A \cap Y) \tag{11.15}$$

is a measure on (X, Σ).

Proof. Direct consequence of Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection (with $\mathrm{card}(I) = 2$)), **properties of intersection**, and Definition 611 (*measure*). □

Remark 630. Note that measures μ_Y and μ'_Y from the two previous lemmas are distinct since they are not defined on the same σ-algebra. But they coincide on the trace σ-algebra $\Sigma \bar{\cap} Y$.

11.2 Negligible subset

Definition 631 ***(negligible subset).*** Let (X, Σ, μ) be a measure space. A subset A of X is said *(μ-)negligible* iff there exists $B \in \Sigma$ such that $A \subset B$ and $\mu(B) = 0$.
The set of μ-negligible subsets is denoted $\mathbf{N}(X, \Sigma, \mu)$ (or simply $\mathbf{N}$).

Definition 632 ***(complete measure).*** Let (X, Σ, μ) be a measure space. The measure μ is said *complete* iff $\mathbf{N}(X, \Sigma, \mu) \subset \Sigma$. If so, the measure space (X, Σ, μ) is also said *complete.*

Definition 633 ***(considerable subset).***
Let (X, Σ, μ) be a measure space. A subset A of X is said *(μ-)considerable* iff $A \notin \mathbf{N}(X, \Sigma, \mu)$.

Lemma 634 ***(equivalent definition of considerable subset).*** *Let (X, Σ, μ) be a measure space. Let $A \subset X$. Then, A is μ-considerable iff for all $B \in \Sigma$, $A \subset B$ implies $\mu(B) > 0$.*

Proof. Direct consequence of Definition 633 (*considerable subset*), Definition 631 (*negligible subset*), Definition 611 (*measure*, nonnegativeness), and **the tautology $\neg(P \wedge Q) \Leftrightarrow (P \Rightarrow \neg Q)$**. □

Remark 635. In the previous lemma, "considerable" naturally means "non-negligible".

Lemma 636 ***(negligibility of measurable subset).***
Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Then, we have $A \in \mathbf{N}$ iff $\mu(A) = 0$.

Proof. Direct consequence of Definition 631 (*negligible subset*, with $B = A$). □

Lemma 637 ***(empty set is negligible).*** *Let (X, Σ, μ) be a measure space. Then, $\varnothing \in \mathbf{N}$.*

Proof. Direct consequence of Lemma 636 (*negligibility of measurable subset*), Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $\varnothing \in \Sigma$), and Definition 611 (*measure*, $\mu(\varnothing) = 0$). □

Lemma 638 ***(compatibility of null measure with countable union).***
Let (X, Σ, μ) be a measure space. Let $I \subset \mathbb{N}$. For all $i \in I$, let $A_i \in \Sigma$ such that $\mu(A_i) = 0$. Then, we have $\mu\left(\bigcup_{i \in I} A_i\right) = 0$.

Proof. Direct consequence of Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, closedness under countable union), Definition 611 (*measure*, nonnegativeness), Lemma 615 (*measure satisfies the finite Boole inequality*), Lemma 620 (*measure satisfies the Boole inequality*), **additive group properties of $\mathbb{R}$**, and Lemma 34 (*stationary sequence is convergent*, countable sum of zero terms is zero). □

Lemma 639 ***($\mathbf{N}$ is closed under countable union).***
Let (X, Σ, μ) be a measure space. Let $I \subset \mathbb{N}$. Let $(A_i)_{i \in I} \in \mathbf{N}$. Then, we have $\bigcup_{i \in I} A_i \in \mathbf{N}$.

Proof. Direct consequence of Definition 631 (*negligible subset*), **monotonicity of union**, and Lemma 638 (*compatibility of null measure with countable union*). □

Lemma 640 ***(subset of negligible is negligible).***
Let (X, Σ, μ) be a measure space. Let $A \in \mathbf{N}$. Then, we have $\mathcal{P}(A) \subset \mathbf{N}$.

Proof. Direct consequence of Definition 631 (*negligible subset*), **the definition of the power set**, and **transitivity of the inclusion**. □

Definition 641 ***(property almost satisfied).***
Let (X, Σ, μ) be a measure space. A predicate P defined on X is said *satisfied (μ-)almost everywhere* iff $\{\neg P\} \stackrel{\text{def.}}{=} \{x \in X \mid \neg P(x)\} \in \mathbf{N}$; this is denoted either "$P\,\mu\,$a.e.", "$\forall x \in X$, $P(x)\,\mu\,$a.e.", "$\forall_{\mu\,\text{a.e.}} x \in X$, $P(x)$", or "for μ-almost all $x \in X$, $P(x)$" (or simply without the mention of the measure μ).

Remark 642. When a single binary relation (using infix notation) is involved in P, the annotation "$\mu\,$a.e." may be put above the infix operator, as in "$\stackrel{\mu\,\text{a.e.}}{=}$" (or simply as in "$\stackrel{\text{a.e.}}{=}$").

Lemma 643 ***(everywhere implies almost everywhere).***
Let (X, Σ, μ) be a measure space. Let P be a predicate on X. Then, we have

$$(\forall x \in X,\ P(x)) \quad \Longrightarrow \quad P\,\mu\,\text{a.e.}. \tag{11.16}$$

Proof. Direct consequence of Definition 641 (*property almost satisfied*), and Lemma 637 (*empty set is negligible*). □

Lemma 644 ***(everywhere implies almost everywhere for almost the same).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let P be a predicate on Y, lifted into a predicate on Y^X. Let f and g be functions from X to Y. Assume that $f \stackrel{\mu\,\text{a.e.}}{=} g$. Then, we have

$$(\forall x \in X,\ P(f(x))) \quad \Longrightarrow \quad P(g)\,\mu\,\text{a.e.} \tag{11.17}$$

Proof. Direct consequence of **monotonicity of complement** $(\{P(g)\}^c \subset \{f = g\}^c)$, Definition 641 (*property almost satisfied*, $\underline{\{f = g\}^c \in \mathbf{N}}$), and Lemma 640 (*subset of negligible is negligible*, $\underline{\{P(g)\}^c \in \mathbf{N}}$). □

Remark 645. The previous lemma allows the use in most statements of functions defined almost everywhere rather than regular total functions.

Lemma 646 ***(extended almost modus ponens).***
Let (X, Σ, μ) be a measure space. Let P and Q be predicates on X. Then, we have

$$(P \stackrel{\mu\,\text{a.e.}}{\Longrightarrow} Q) \quad \wedge \quad P\,\mu\,\text{a.e.} \quad \Longrightarrow \quad Q\,\mu\,\text{a.e.} \tag{11.18}$$

Proof. Assume that $P \Rightarrow Q$ and P hold almost everywhere.
From Definition 641 (*property almost satisfied*), and **modus ponens**, we have

$$\{P \Rightarrow Q\}^c, \{P\}^c \in \mathbf{N} \quad \text{and} \quad B \stackrel{\text{def.}}{=} \{P \Rightarrow Q\} \cap \{P\} \subset \{Q\}.$$

Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 639 (**N** *is closed under countable union*, $\underline{\text{with } \mathrm{card}(I) = 2}$), we have

$$\{Q\}^c \subset B^c \quad \text{and} \quad B^c = \{P \Rightarrow Q\}^c \cup \{P\}^c \in \mathbf{N}.$$

Therefore, from Lemma 640 (*subset of negligible is negligible*), and Definition 641 (*property almost satisfied*), we have for μ-almost all $x \in X$, $Q(x)$. □

Lemma 647 ***(almost modus ponens).***
Let (X, Σ, μ) be a measure space. Let P and Q be predicates on X. Then, we have

$$(\forall x \in X,\ P(x) \Rightarrow Q(x)) \quad \wedge \quad P\,\mu\,\text{a.e.} \quad \Longrightarrow \quad Q\,\mu\,\text{a.e.} \tag{11.19}$$

Proof. Direct consequence of Lemma 643 (*everywhere implies almost everywhere*, $\underline{\text{with predicate } P \Rightarrow Q}$), and Lemma 646 (*extended almost modus ponens*). □

Remark 648. The two previous lemmas allow to still use modus ponens in reasoning when predicates are only valid almost everywhere.

Definition 649 ***(almost definition).***
Let (X, Σ, μ) be a measure space. Let Y be a set. A function $f : X \to Y$ is said *defined (μ-)almost everywhere in* X iff the property "$f(x)$ is defined" is satisfied μ-almost everywhere.

The *set of functions* $X \to Y$ *defined (μ-)almost everywhere* is denoted $\left(Y^X\right)_{\mu\,\text{a.e.}}$, or through the type annotation $(X \to Y)_{\mu\,\text{a.e.}}$ (or simply without the mention of the measure μ).

Definition 650 ***(almost binary relation).*** Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. The functions $f, g : (X \to Y)_{\mu\,\text{a.e.}}$ are said *in relation (μ-)almost everywhere through* $\mathcal{R}$ iff the property "$f(x)\ \mathcal{R}\ g(x)$" is satisfied μ-almost everywhere; this is denoted $f\ \mathcal{R}_{\mu\,\text{a.e.}}\ g$ (or simply $f\ \mathcal{R}_{\text{a.e.}}\ g$).

Lemma 651 ***(compatibility of almost binary relation with reflexivity).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is reflexive. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is also reflexive.

Proof. Direct consequence of Definition 650 (*almost binary relation*), Definition 649 (*almost definition*), Definition 641 (*property almost satisfied*), **the definition of reflexivity**, Lemma 643 (*everywhere implies almost everywhere*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*). $\square$

Lemma 652 ***(compatibility of almost binary relation with symmetry).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is symmetric. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is also symmetric.

Proof. Direct consequence of Definition 650 (*almost binary relation*), Definition 649 (*almost definition*), Definition 641 (*property almost satisfied*), **the definition of symmetry**, Lemma 647 (*almost modus ponens*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*). $\square$

Lemma 653 ***(compatibility of almost binary relation with antisymmetry).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is antisymmetric. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is "almost" antisymmetric (where equality is replaced by almost equality).

Proof. Let $f, g : (X \to Y)_{\mu\,\text{a.e.}}$. Assume that $f\ \mathcal{R}_{\mu\,\text{a.e.}}\ g$ and $g\ \mathcal{R}_{\mu\,\text{a.e.}}\ f$. Let us show that $f \overset{\mu\,\text{a.e.}}{=} g$.

From Definition 650 (*almost binary relation*), Definition 649 (*almost definition*), Definition 641 (*property almost satisfied*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*), we have $\{f\ \mathcal{R}\ g\}^c, \{g\ \mathcal{R}\ f\}^c \in \mathbf{N}$. Moreover, from **the definition of antisymmetry**, we have $B \overset{\text{def.}}{=} \{f\ \mathcal{R}\ g\} \cap \{g\ \mathcal{R}\ f\} \subset \{f = g\}$. Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 639 (**N** *is closed under countable union*, with $\text{card}(I) = 2$), we have $\{f = g\}^c \subset B^c = \{f\ \mathcal{R}\ g\}^c \cup \{g\ \mathcal{R}\ f\}^c \in \mathbf{N}$.

Therefore, from Lemma 640 (*subset of negligible is negligible*), and Definition 641 (*property almost satisfied*), we have $f \overset{\mu\,\text{a.e.}}{=} g$. $\square$

Lemma 654 ***(compatibility of almost binary relation with transitivity).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is transitive. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is also transitive.

Proof. Let $f, g, h : (X \to Y)_{\mu\,\text{a.e.}}$. Assume that $f\ \mathcal{R}_{\mu\,\text{a.e.}}\ g$ and $g\ \mathcal{R}_{\mu\,\text{a.e.}}\ h$. Let us show that $f\ \mathcal{R}_{\mu\,\text{a.e.}}\ h$.

From Definition 650 (*almost binary relation*), Definition 649 (*almost definition*), Definition 641 (*property almost satisfied*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*), we have $\{f \mathcal{R} g\}^c, \{g \mathcal{R} h\}^c \in \mathbf{N}$. Moreover, from **the definition of transitivity**, we have $B \stackrel{\text{def.}}{=} \{f \mathcal{R} g\} \cap \{g \mathcal{R} h\} \subset \{f \mathcal{R} h\}$. Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 639 (**N** *is closed under countable union*, with $\operatorname{card}(I) = 2$), we have $\{f \mathcal{R} h\}^c \subset B^c = \{f \mathcal{R} g\}^c \cup \{g \mathcal{R} h\}^c \in \mathbf{N}$.

Therefore, from Lemma 640 (*subset of negligible is negligible*), Definition 641 (*property almost satisfied*), and Definition 650 (*almost binary relation*), we have $f \mathcal{R}_{\mu\,\text{a.e.}}\, h$. □

Remark 655. From the similarity of their formal expressions, antisymmetry and transitivity can be abstracted under the form

$$\forall f, g, h : X \to Y, \quad f \mathcal{R} g \quad \wedge \quad g \mathcal{R} F(f, h) \quad \Longrightarrow \quad f \mathcal{R}' F(g, h)$$

where the binary relation $\mathcal{R}'$ is the equality for antisymmetry and $\mathcal{R}$ for transitivity, and where the function $F : Y^X \times Y^X \to Y^X$ is the first projection for antisymmetry and the second projection for transitivity. Moreover, the proofs of the two previous lemmas only differ from the expression of their instances of $\mathcal{R}'$ and F. Thus, both statements and proofs of these lemmas can be abstracted under more general forms.

Lemma 656 ***(almost equivalence is equivalence relation).*** *Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is an equivalence relation. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is also an equivalence relation.*

Proof. Direct consequence of **the definition of equivalence relation**, Lemma 651 (*compatibility of almost binary relation with reflexivity*), Lemma 652 (*compatibility of almost binary relation with symmetry*), and Lemma 654 (*compatibility of almost binary relation with transitivity*). □

Lemma 657 ***(almost equality is equivalence relation).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Then, $\stackrel{\mu\,\text{a.e.}}{=}$ is an equivalence relation over Y^X.

Proof. Direct consequence of Lemma 656 (*almost equivalence is equivalence relation*, with equality over Y). □

Lemma 658 ***(almost order is order relation).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $\mathcal{R}$ be a binary relation over Y, lifted into a binary relation over Y^X. Assume that $\mathcal{R}$ is an order relation. Then, $\mathcal{R}_{\mu\,\text{a.e.}}$ is an "almost" order relation (where equality is replaced by almost equality in antisymmetry).

Proof. Direct consequence of **the definition of order relation**, Lemma 651 (*compatibility of almost binary relation with reflexivity*), Lemma 653 (*compatibility of almost binary relation with antisymmetry*), and Lemma 654 (*compatibility of almost binary relation with transitivity*). □

Lemma 659 ***(compatibility of almost binary relation with operator).***
Let (X, Σ, μ) be a measure space. Let Y be a nonempty set. Let $y_0 \in Y$. Let $\mathcal{R}$ and $\mathcal{R}'$ be binary relations over Y, lifted into binary relations over Y^X. Assume that $y_0 \mathcal{R} y_0$. Let I be a nonempty subset of $\mathbb{N}$. Let $\Diamond : Y^I \to Y$, lifted into an operator $(Y^X)^I \to Y^X$. Assume that for all $(f_i)_{i \in I}, (g_i)_{i \in I} : X \to Y$, we have

$$(\forall i \in I,\ f_i \mathcal{R} g_i) \quad \Longrightarrow \quad \Diamond(f_i)_{i \in I} \mathcal{R}' \Diamond(g_i)_{i \in I}. \tag{11.20}$$

Then, for all $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \to Y)_{\mu\,\text{a.e.}}$, we have

$$(\forall i \in I,\ f_i \mathcal{R}_{\mu\,\text{a.e.}}\, g_i) \quad \Longrightarrow \quad \Diamond(f_i)_{i \in I} \mathcal{R}'_{\mu\,\text{a.e.}} \Diamond(g_i)_{i \in I}. \tag{11.21}$$

Proof. For all $i \in I$, let $f_i, g_i : (X \to Y)_{\mu\text{ a.e.}}$, and assume that $f_i\ \mathcal{R}_{\mu\text{ a.e.}}\ g_i$ holds.

For all $i \in I$, let $B_i \overset{\text{def.}}{=} \{f_i\ \mathcal{R}\ g_i\}$. For all $i \in I$, let $\tilde{f}_i, \tilde{g}_i : X \to Y$ defined by

$$\left.\begin{array}{l} \tilde{f}_i(x) \overset{\text{def.}}{=} f_i(x) \\ \tilde{g}_i(x) \overset{\text{def.}}{=} g_i(x) \end{array}\right\} \text{ when } x \in \bigcap_{i \in I} B_i, \quad \text{and} \quad \tilde{f}_i(x) = \tilde{g}_i(x) \overset{\text{def.}}{=} y_0 \text{ otherwise.}$$

Let $i \in I$. Then, from Definition 650 (*almost binary relation*), Definition 649 (*almost definition*), Definition 641 (*property almost satisfied*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*), we have $B_i^c \in \mathbf{N}$. Let $x \in X$. Then, by construction, and from reflexivity of $\mathcal{R}$, we have $\tilde{f}_i(x)\ \mathcal{R}\ \tilde{g}_i(x)$. Thus, from assumption, $\Diamond(\tilde{f}_i)_{i \in I}(x)\ \mathcal{R}'\ \Diamond(\tilde{g}_i)_{i \in I}(x)$ also holds. Moreover, by construction, we have

$$\forall x \in \bigcap_{i \in I} B_i, \quad \Diamond(\tilde{f}_i)_{i \in I}(x) = \Diamond(f_i)_{i \in I}(x) \quad \text{and} \quad \Diamond(\tilde{g}_i)_{i \in I}(x) = \Diamond(g_i)_{i \in I}(x).$$

Thus, we have $\bigcap_{i \in I} B_i \subset \{\Diamond(f_i)_{i \in I}\ \mathcal{R}'\ \Diamond(g_i)_{i \in I}\}$. Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 639 ($\mathbf{N}$ *is closed under countable union*), we have

$$\{\Diamond(f_i)_{i \in I}\ \mathcal{R}'\ \Diamond(g_i)_{i \in I}\}^c \subset \bigcup_{i \in I} \{f_i\ \mathcal{R}\ g_i\}^c \in \mathbf{N}.$$

Therefore, from Definition 641 (*property almost satisfied*), Definition 650 (*almost binary relation*), and Definition 649 (*almost definition*), we have $\Diamond(f_i)_{i \in I}\ \mathcal{R}'_{\mu\text{ a.e.}}\ \Diamond(g_i)_{i \in I}$. □

Lemma 660 ***(compatibility of almost equality with operator).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Let $I \subset \mathbb{N}$. Let $\Diamond : Y^I \to Y$, lifted into an operator $(Y^X)^I \to Y^X$. Let $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \to Y)_{\mu\text{ a.e.}}$. Assume that for all $i \in I$, $f_i \overset{\mu\text{ a.e.}}{=} g_i$. Then, we have $\Diamond(f_i)_{i \in I} \overset{\mu\text{ a.e.}}{=} \Diamond(g_i)_{i \in I}$.

Proof. Direct consequence of Lemma 659 (*compatibility of almost binary relation with operator*, with $\mathcal{R} = \mathcal{R}' \overset{\text{def.}}{=}$ equality), **reflexivity of equality**, and **the definition of function ((for all $i \in I$, $f_i = g_i$) implies $\Diamond(f_i)_{i \in I} = \Diamond(g_i)_{i \in I}$)**. □

Lemma 661 ***(compatibility of almost inequality with operator).***
Let (X, Σ, μ) be a measure space. Let Y be an ordered set. Let $\mathcal{R}'$ be a binary relation on Y, lifted into a binary relation on Y^X. Let $I \subset \mathbb{N}$. Let $\Diamond : Y^I \to Y$, lifted into an operator $(Y^X)^I \to Y^X$. Assume that for all $(f_i)_{i \in I}, (g_i)_{i \in I} : X \to Y$, we have

$$(\forall i \in I,\ f_i \leqslant g_i) \implies \Diamond(f_i)_{i \in I}\ \mathcal{R}'\ \Diamond(g_i)_{i \in I}. \tag{11.22}$$

Then, for all $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \to Y)_{\mu\text{ a.e.}}$, we have

$$(\forall i \in I,\ f_i \overset{\mu\text{ a.e.}}{\leqslant} g_i) \implies \Diamond(f_i)_{i \in I}\ \mathcal{R}'_{\mu\text{ a.e.}}\ \Diamond(g_i)_{i \in I}. \tag{11.23}$$

Proof. Direct consequence of Lemma 659 (*compatibility of almost binary relation with operator*, with $\mathcal{R} \overset{\text{def.}}{=} \leqslant$), and **reflexivity of inequality**. □

Remark 662. The previous generic results apply to unary operators $\Diamond$ such as the scalar multiplication, or the absolute value (for which $\mathrm{card}(I) = 1$), as well as to binary (or n-ary) operators $\Diamond$ such as the addition, the multiplication, the maximum, or the minimum (for which $\mathrm{card}(I) \geqslant 2$ is finite). But it also applies to operators $\Diamond$ taking a countable number of arguments such as the infimum, the supremum, or the limit of a pointwise convergent sequence.

Remark 663. Note that the previous lemma may be useful with $\mathcal{R}'$ distinct from $\leqslant$. For instance, when the operator $\Diamond$ is the scalar multiplication by a negative number, we need $\mathcal{R}' =\geqslant$.

Lemma 664 ***(definiteness implies almost definiteness).***
Let (X, Σ, μ) be a measure space. Let Y be a set. Assume that $0 \in Y$. Let $\Diamond : Y \to Y$, lifted into an operator $Y^X \to Y^X$. Assume that $\Diamond$ is definite:

$$\forall y \in Y, \quad \Diamond(y) = 0 \quad \Longrightarrow \quad y = 0. \tag{11.24}$$

Then, $\Diamond$ is also "almost definite":

$$\forall f \in \left(Y^X\right)_{\mu\,\text{a.e.}}, \quad \Diamond(f) \overset{\mu\,\text{a.e.}}{=} 0 \quad \Longrightarrow \quad f \overset{\mu\,\text{a.e.}}{=} 0. \tag{11.25}$$

Proof. Direct consequence of Lemma 647 (*almost modus ponens*), and Lemma 644 (*everywhere implies almost everywhere for almost the same*). $\square$

Remark 665. Note that Lemma 660 provides the implication in the other direction.

11.3 Uniqueness condition

Remark 666. The next lemma is useful to establish in Section 12.2 uniqueness of the Lebesgue measure that generalizes the length of bounded intervals (see Theorem 724).

Remark 667. The next proof follows the Dynkin π–λ theorem scheme (see Section 4.3).

Lemma 668 ***(uniqueness of measures extended from a π-system).***
Let (X, Σ) be a measurable space. Let μ_1 and μ_2 be measures on (X, Σ). Let $G \subset \mathcal{P}(X)$. Assume that G is a π-system, a generator of Σ, contains a countable pseudopartition $(X_n)_{n\in\mathbb{N}}$ of X with finite measure μ_1, and that both measures coincide on G. Then, we have $\mu_1 = \mu_2$.

Proof. Let $A \in G$. Assume that $\mu_1(A) < \infty$ (e.g. $A \stackrel{\text{def.}}{=} X_0$).
Let $B \in \Sigma$. Then, from Lemma 483 (*generated σ-algebra is minimum*, $A \in G \subset \Sigma$), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection with card(I) equals 2), we have $B \cap A \in \Sigma$. Let $\mathcal{S}_A \stackrel{\text{def.}}{=} \{B \in \Sigma \,|\, \mu_1(B \cap A) = \mu_2(B \cap A)\}$.

(1). $G \neq \varnothing \;\wedge\; \Pi_X(G) \subset \mathcal{S}_A$.
Let $B \in G$. Then, from Definition 428 (*π-system*, closedness under intersection), $B \cap A \in G$, i.e. $B \in \mathcal{S}_A$. Thus, we have $G \subset \mathcal{S}_A$. Hence, from Definition 428 (*π-system*, nonemptiness), and Lemma 435 (*π-system generation is idempotent*), we have $G \neq \varnothing$ and $\Pi_X(G) = G \subset \mathcal{S}_A$.

(2). $\mathcal{S}_A$ is λ-system.
From **the identity $X \cap A = A$**, and Lemma 475 (*equivalent definition of σ-algebra*, contains the full set), we have $X \in \mathcal{S}_A$.
Let $B_1, B_2 \in \mathcal{S}_A$. Assume that $B_1 \subset B_2$. Then, from **distributivity of intersection over local complement**, and Lemma 614 (*measure is monotone*, with μ_1 and μ_2), we have

$$\mu_2(B_1 \cap A) = \mu_1(B_1 \cap A) \leqslant \mu_1(A) < \infty, \qquad \text{and}$$
$$\begin{aligned}\mu_1((B_2 \setminus B_1) \cap A) &= \mu_1(B_2 \cap A \setminus B_1 \cap A) = \mu_1(B_2 \cap A) - \mu_1(B_1 \cap A)\\ &= \mu_2(B_2 \cap A) - \mu_2(B_1 \cap A) = \mu_2(B_2 \cap A \setminus B_1 \cap A) = \mu_2((B_2 \setminus B_1) \cap A).\end{aligned}$$

Then, $B_2 \setminus B_1 \in \mathcal{S}_A$. Thus, $\mathcal{S}_A$ is closed under local complement.
Let $(B_n)_{n\in\mathbb{N}} \in \mathcal{S}_A$. Assume that for all $n \in \mathbb{N}$, $B_n \subset B_{n+1}$. Then, from **distributivity of intersection over union**, Lemma 617 (*measure is continuous from below*, with μ_1 and μ_2, and nondecreasing $(B_n \cap A)_{n\in\mathbb{N}}$), and Definition 616 (*continuity from below*), we have

$$\begin{aligned}\mu_1\left(\bigcup_{n\in\mathbb{N}} B_n \cap A\right) &= \mu_1\left(\bigcup_{n\in\mathbb{N}} (B_n \cap A)\right) = \lim_{n\to\infty} \mu_1(B_n \cap A)\\ &= \lim_{n\to\infty} \mu_2(B_n \cap A) = \mu_2\left(\bigcup_{n\in\mathbb{N}} (B_n \cap A)\right) = \mu_2\left(\bigcup_{n\in\mathbb{N}} B_n \cap A\right).\end{aligned}$$

Then, $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{S}_A$. Thus, $\mathcal{S}_A$ is closed under countable monotone union.
Hence, from Lemma 460 (*equivalent definition of λ-system*), $\mathcal{S}_A$ is a λ-system on X.

(3). $\mathcal{S}_A = \Sigma$. Direct consequence of (1), (2), and Lemma 510 (*usage of Dynkin π–λ theorem*, with P being the function $B \mapsto \mu_1(B \cap A) = \mu_2(B \cap A)$).
Hence, for all $B \in \Sigma$, for all $A \in G$ such that $\mu_1(A) < \infty$, we have $\mu_1(B \cap A) = \mu_2(B \cap A)$.

(4). $\mu_1 = \mu_2$. Let $B \in \Sigma$.
For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} B \cap X_n$. Then, from Definition 207 (*pseudopartition*), Lemma 483 (*generated σ-algebra is minimum*, $X_n \in G \subset \Sigma$), Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection with card(I) equals 2), **compatibility of intersection**

with pairwise disjunction, **left distributivity of intersection over union**, and **identity law for intersection**, $(B_n)_{n\in\mathbb{N}} \in \Sigma$ is pairwise disjoint, and we have $\biguplus_{n\in\mathbb{N}} B_n = B$.

Let $n \in \mathbb{N}$. Then, from (3) (with $B \in \Sigma$, and $A \stackrel{\text{def.}}{=} X_n \in G$ such that $\mu_1(X_n) < \infty$), we have $\mu_1(B_n) = \mu_2(B_n)$. Hence, from Definition 611 (*measure*, μ_1 and μ_2 are σ-additive), and Definition 608 (*σ-additivity over measurable space*), we have

$$\mu_1(B) = \sum_{n\in\mathbb{N}} \mu_1(B_n) = \sum_{n\in\mathbb{N}} \mu_2(B_n) = \mu_2(B).$$

Therefore, we have the equality $\mu_1 = \mu_2$. □

11.4 Some measures

Lemma 669 ***(trivial measure).***
Let (X, Σ) be a measurable space. Then, the zero function on Σ is a measure on (X, Σ).
It is called the trivial measure *on (X, Σ), and $(X, \Sigma, 0)$ is called the* trivial measure space.

Proof. Direct consequence of Definition 611 (*measure*), and **ordered group properties of $\mathbb{R}$**. □

Lemma 670 ***(equivalent definition of trivial measure).***
Let (X, Σ, μ) be a measure space. Then, μ is the trivial measure iff $\mu(X) = 0$.

Proof. Direct consequence of Lemma 669 (*trivial measure*), Definition 611 (*measure*, μ is nonnegative), and Lemma 614 (*measure is monotone*, μ is nonpositive). □

Lemma 671 ***(counting measure).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Let δ_Y be the function defined by

$$\forall A \in \Sigma, \quad \delta_Y(A) \stackrel{\text{def.}}{=} \operatorname{card}(A \cap Y) \tag{11.26}$$

with the convention that the cardinality of an infinite set is ∞. Then, δ_Y is a measure on (X, Σ).
It is called the counting measure (associated with Y).

Proof. From **the definition of the cardinality**, and **the definition of ∞**, the function δ_Y is nonnegative. Moreover, from Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $\varnothing \in \Sigma$), and **the definition of $\varnothing$**, we have

$$\delta_Y(\varnothing) = \operatorname{card}(\varnothing) = 0.$$

Let $I \subset \mathbb{N}$. Let $(A_i)_{i \in I} \in \Sigma$. Assume that the A_i's are pairwise disjoint. Then, from **distributivity of intersection over disjoint union**, and **σ-additivity of the cardinality (with the convention that ∞ is absorbing element for addition in $\overline{\mathbb{N}}$)**, we have

$$\delta_Y\left(\biguplus_{i \in I} A_i\right) = \operatorname{card}\left(\biguplus_{i \in I} A_i \cap Y\right) = \operatorname{card}\left(\biguplus_{i \in I} (A_i \cap Y)\right) = \sum_{i \in I} \operatorname{card}(A_i \cap Y) = \sum_{i \in I} \delta_Y(A_i).$$

Therefore, from Definition 608 (*σ-additivity over measurable space*), and Definition 611 (*measure*), δ_Y is a measure on (X, Σ). □

Lemma 672 ***(finiteness of counting measure).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Then, δ_Y is finite iff the set Y is finite.

Proof. Direct consequence of Lemma 475 (*equivalent definition of σ-algebra*, contains full set), Lemma 671 (*counting measure*, $\delta_Y(X) = \operatorname{card}(Y)$), and Definition 622 (*finite measure*). □

Lemma 673 ***(σ-finite counting measure).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Assume that Σ contains all singletons of X. Then, δ_Y is σ-finite implies Y is countable, and X is countable implies δ_Y is σ-finite.

Proof. From Definition 611 (*measure*), Definition 516 (*measurable space*), and Definition 474 (*σ-algebra*, closedness under countable union), the σ-algebra Σ contains all countable subsets of X. Hence, if X is countable, we have $\Sigma = \mathcal{P}(X)$.

First implication. Assume that δ_Y is σ-finite.
Then, from Lemma 625 (*equivalent definition of σ-finite measure*), let $(B_n)_{n \in \mathbb{N}} \in \Sigma$ such that for all $n \in \mathbb{N}$, $B_n \subset B_{n+1}$, $\delta_Y(B_n) = \operatorname{card}(B_n \cap Y) < \infty$, and $X = \bigcup_{n \in \mathbb{N}} B_n$. For all $n \in \mathbb{N}$,

let $A_n \overset{\text{def.}}{=} B_n \cap Y$. Let $n \in \mathbb{N}$. Then, from **De Morgan's laws**, and Lemma 671 (*counting measure*), we have $Y = \bigcup_{n\in\mathbb{N}} A_n$, and $\operatorname{card}(A_n) = \delta_Y(B_n) < \infty$. Hence, from **countability of countable union of finite subsets**, Y is countable.

Second implication. Assume that X is countable. Then, we have $\Sigma = \mathcal{P}(X)$.
From **the definition of countability**, let φ be a bijection from $I \subset \mathbb{N}$ onto X. For all $n \in \mathbb{N}$, let $A_n \overset{\text{def.}}{=} \varphi([0..n] \cap I) \in \Sigma$. Let $n \in \mathbb{N}$. Then, from Lemma 671 (*counting measure*), **monotonicity of the cardinality**, **preservation of the cardinality by bijection**, **compatibility of direct image with union**, and **De Morgan's laws**, we have

$$\delta_Y(A_n) = \operatorname{card}(A_n \cap Y) \leqslant \operatorname{card}(A_n) = \operatorname{card}([0..n] \cap I) \leqslant n + 1 < \infty,$$

$$\bigcup_{n\in\mathbb{N}} A_n = \varphi\left(\bigcup_{n\in\mathbb{N}} ([0..n] \cap I)\right) = \varphi\left(\bigcup_{n\in\mathbb{N}} [0..n] \cap I\right) = \varphi(I) = X.$$

Hence, from Definition 624 (*σ-finite measure*), the counting measure δ_Y is σ-finite.

Therefore, we have both implications. □

Remark 674. Note that the set Y is not required to be measurable. Note also that the previous lemmas are valid for any σ-algebra Σ, including the discrete σ-algebra $\mathcal{P}(X)$.
Counting measures are usually considered in the case where $Y = X$ is countable, and Σ is the discrete σ-algebra $\mathcal{P}(X)$. A typical example is the σ-finite measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \delta_{\mathbb{N}})$.

Definition 675 ***(Dirac measure).***
Let (X, Σ) be a measurable space. Let $a \in X$. The counting measure associated with $\{a\}$ is called *Dirac measure (at a)*; it is also denoted $\delta_a \overset{\text{def.}}{=} \delta_{\{a\}}$.

Lemma 676 ***(equivalent definition of Dirac measure).***
Let (X, Σ) be a measurable space. Let $a \in X$, and let $A \in \Sigma$. Then, we have $\delta_a(A) = \mathbb{1}_A(a)$.

Proof. Direct consequence of Definition 675 (*Dirac measure*), and Lemma 671 (*counting measure*, $\operatorname{card}(A \cap \{a\}) = 1$ when $a \in A$ and 0 otherwise). □

Lemma 677 ***(Dirac measure is finite).***
Let (X, Σ) be a measurable space. Let $a \in X$. Then, δ_a is finite and $\delta_a(X) = 1$.

Proof. Direct consequence of Definition 675 (*Dirac measure*), Lemma 672 (*finiteness of counting measure*, δ_a is finite), and Lemma 671 (*counting measure*, $\operatorname{card}(X \cap \{a\}) = 1$). □

Chapter 12

Measure and numbers

Contents

12.1 Negligibility and numbers

Definition 678 ***(summability domain).*** Let (X, Σ, μ) be a measure space. Let $f, g : X \to \overline{\mathbb{R}}$. The *summability domain of f and g* is denoted $\mathcal{D}^+(f, g)$; it is defined by

$$\mathcal{D}^+(f,g) \stackrel{\text{def.}}{=} [(\{f = \infty\} \cap \{g = -\infty\}) \cup (\{f = -\infty\} \cap \{g = \infty\})]^c. \tag{12.1}$$

Lemma 679 ***(summability on summability domain).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Then, $f(x)+g(x)$ is well-defined iff $x \in \mathcal{D}^+(f, g)$.

Proof. Direct consequence of Definition 282 (*addition in $\overline{\mathbb{R}}$*), and Definition 678 (*summability domain*). □

Lemma 680 ***(measurability of summability domain).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Then, $\mathcal{D}^+(f, g) \in \Sigma$.

Proof. Direct consequence of Lemma 579 (*inverse image is measurable*), and Lemma 475 (*equivalent definition of σ-algebra*)closedness under countable intersection, union and complement. □

Lemma 681 ***(negligibility of summability domain).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Then, $f + g$ is well-defined almost everywhere iff $[\mathcal{D}^+(f, g)]^c \in \mathbf{N}$.*

Proof. Direct consequence of Lemma 679 (*summability on summability domain*), and Definition 641 (*property almost satisfied*), □

Lemma 682 ***(almost sum).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Let $A \stackrel{\text{def.}}{=} \mathcal{D}^+(f, g)$. Let $\tilde{f} \stackrel{\text{def.}}{=} f\mathbb{1}_A$ and $\tilde{g} \stackrel{\text{def.}}{=} g\mathbb{1}_A$. Assume that $f+g$ is well-defined almost everywhere. Then, we have $\tilde{f} \stackrel{\mu\,\text{a.e.}}{=} f$, $\tilde{g} \stackrel{\mu\,\text{a.e.}}{=} g$, $\mathcal{D}^+(\tilde{f}, \tilde{g}) = X$, and $\tilde{f}, \tilde{g}, \tilde{f} + \tilde{g} \in \mathcal{M}$.*
The sum $\tilde{f} + \tilde{g}$ is called the almost sum of f and g; *it is denoted $f \stackrel{\mu\,\text{a.e.}}{+} g$.*

Proof. From Lemma 681 (*negligibility of summability domain*), and Definition 641 (*property almost satisfied*, with $\{\tilde{f} \neq f\} = \{\tilde{g} \neq g\} = A^c$), we have $\tilde{f} \stackrel{\mu\,\text{a.e.}}{=} f$ and $\tilde{g} \stackrel{\mu\,\text{a.e.}}{=} g$. From Lemma 680 (*measurability of summability domain*), Lemma 569 (*measurability of indicator function*), and Lemma 583 (*$\mathcal{M}$ is closed under multiplication*), we have $\tilde{f}, \tilde{g} \in \mathcal{M}$. Hence, from Definition 282 (*addition in $\overline{\mathbb{R}}$*), and Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*), $\tilde{f} + \tilde{g}$ is well-defined, and we have $\tilde{f} + \tilde{g} \in \mathcal{M}$. □

Lemma 683 *(compatibility of almost sum with almost equality).* *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f + g$ is well-defined almost everywhere. Let $f', g' \in \mathcal{M}$. Assume that $f' \overset{\mu\,\text{a.e.}}{=} f$, $g' \overset{\mu\,\text{a.e.}}{=} g$, and $\mathcal{D}^+(f', g') = X$. Then, $f' + g' \in \mathcal{M}$, and we have $f' + g' \overset{\mu\,\text{a.e.}}{=} \tilde{f} \overset{\mu\,\text{a.e.}}{+} \tilde{g}$.*

Proof. From Lemma 637 (*empty set is negligible*), Lemma 681 (*negligibility of summability domain*), and Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*), $f' + g'$ is well-defined, and we have $f' + g' \in \mathcal{M}$. let $A \overset{\text{def.}}{=} \mathcal{D}^+(f, g)$, $\tilde{f} \overset{\text{def.}}{=} f\mathbb{1}_A$ and $\tilde{g} \overset{\text{def.}}{=} g\mathbb{1}_A$. Therefore, from Lemma 657 (*almost equality is equivalence relation*, transitivity), Lemma 660 (*compatibility of almost equality with operator*, with the binary operator addition), and Lemma 682 (*almost sum*), we have $f' \overset{\mu\,\text{a.e.}}{=} \tilde{f}$, $g' \overset{\mu\,\text{a.e.}}{=} \tilde{g}$, and $f' + g' \overset{\mu\,\text{a.e.}}{=} \tilde{f} + \tilde{g} \overset{\mu\,\text{a.e.}}{=} f \overset{\mu\,\text{a.e.}}{+} g$. □

Remark 684. Note that from Lemma 682, such functions f' and g' actually exist in $\mathcal{M}$.

Lemma 685 *(almost sum is sum).* *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $\mathcal{D}^+(f, g) = X$. Then, $f + g \in \mathcal{M}$, and we have $f \overset{\mu\,\text{a.e.}}{+} g = f + g$.*

Proof. Direct consequence of Definition 678 (*summability domain*, $f + g$ is well-defined everywhere), and Lemma 682 (*almost sum*, with $\tilde{f} = f$ and $\tilde{g} = g$). □

Lemma 686 *(absolute value is almost definite).* *Let (X, Σ, μ) be a measure space. Then, the absolute value, lifted into an operator $\overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^X$, is almost definite:*

$$\forall f \in \left(\overline{\mathbb{R}}^X\right)_{\mu\,\text{a.e.}}, \quad |f| \overset{\mu\,\text{a.e.}}{=} 0 \iff f \overset{\mu\,\text{a.e.}}{=} 0. \tag{12.2}$$

Proof. Direct consequence of Lemma 664 (*definiteness implies almost definiteness*, with the unary operator absolute value), Lemma 660 (*compatibility of almost equality with operator*, with the unary operator absolute value), and Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*). □

Lemma 687 *(masking almost nowhere).* *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Let $A \in \Sigma$. Assume that $\mu(A^c) = 0$. Then, we have $f \overset{\mu\,\text{a.e.}}{=} f\mathbb{1}_A$.*

Proof. Direct consequence of Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, closedness under complement), **the definition of the indicator function ($A \subset \{f = f\mathbb{1}_A\}$), monotonicity of complement ($\{f = f\mathbb{1}_A\}^c$ is included in A^c)**, Definition 631 (*negligible subset*). and Definition 641 (*property almost satisfied*). □

Lemma 688 *(finite nonnegative part).* *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Let $A \in \Sigma$. Let $A_f \overset{\text{def.}}{=} A \cap f^{-1}(\mathbb{R}_+)$. Let $\tilde{f}_A \overset{\text{def.}}{=} f\mathbb{1}_{A_f}$. Then, $A_f \in \Sigma$ and $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$.*

Moreover, assume that f is μ-almost everywhere finite and nonnegative, and that $\mu(A^c) = 0$. Then, we have $\mu(A_f^c) = 0$, i.e. $f \overset{\mu\,\text{a.e.}}{=} \tilde{f}_A$.

Proof. Let $B \overset{\text{def.}}{=} f^{-1}(\overline{\mathbb{R}}_+)$ and $C \overset{\text{def.}}{=} f^{-1}(\mathbb{R})$. Then, from **compatibility of inverse image with intersection**, the definition of A_f, and **properties of the indicator function**, we have

$$f^{-1}(\mathbb{R}_+) = B \cap C, \quad A_f = A \cap B \cap C, \quad \text{and} \quad \mathbb{1}_{A_f} = \mathbb{1}_A \mathbb{1}_B \mathbb{1}_C.$$

Hence, from Lemma 400 (*equivalent definition of nonnegative and nonpositive parts*), and Definition 397 (*finite part*), $\tilde{f}_A$ is the finite part of $(f\mathbb{1}_A)^+$. Therefore, from Lemma 591 (*measurability and masking*, $f\mathbb{1}_A \in \mathcal{M}$), Lemma 594 (*measurability of nonnegative and nonpositive parts*, with $f\mathbb{1}_A$), and Lemma 595 ($\mathcal{M}_+$ *is closed under finite part*, with $(f\mathbb{1}_A)^+$), we have $\tilde{f}_A \in \mathcal{M}_\mathbb{R} \cap \mathcal{M}_+$.

Assume now that f is μ-almost everywhere finite and nonnegative, and that $\mu(A^c) = 0$. Then, from Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (σ*-algebra*, closedness under complement), Definition 641 (*property almost satisfied*), and Lemma 636 (*negligibility of measurable subset*), we have

$$B^c, C^c \in \Sigma \quad \text{and} \quad \mu(B^c) = \mu(C^c) = 0.$$

Hence, from Definition 474 (σ*-algebra*, closedness under complement), **De Morgan's laws**, and Lemma 615 (*measure satisfies the finite Boole inequality*), we have

$$A_f^c \in \Sigma \quad \text{and} \quad \mu(A_f^c) \leqslant \mu(A^c) + \mu(B^c) + \mu(C^c) = 0.$$

Therefore, from Definition 611 (*measure*, nonnegativeness), and Lemma 687 (*masking almost nowhere*), we have $\mu(A_f^c) = 0$ and $f \overset{\mu\,\text{a.e.}}{=} \tilde{f}_A$. □

12.2 The Lebesgue measure

Remark 689. This section follows Carathéodory's extension scheme (see Section 4.2).

Remark 690. We recall the notation $[\![\cdot,\cdot]\!]$ for not specifying open or closed bounds for intervals.

Definition 691 ***(length of interval).***
Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. The *length of interval* from a to b is $\ell([\![a,b]\!]) \stackrel{\text{def.}}{=} b - a$.

Lemma 692 ***(length is nonnegative).*** *Let $a, b \in \mathbb{R}$, Assume that $a \leqslant b$. Then, $\ell((a,b)) \geqslant 0$.*

Proof. Direct consequence of Definition 691 (*length of interval*). ☐

Lemma 693 ***(length is homogeneous).*** *We have $\ell(\varnothing) = 0$.*

Proof. Let $a \in \mathbb{R}$. Then, for instance, we have $\varnothing = (a, a)$. Therefore, from Definition 691 (*length of interval*), we have $\ell(\varnothing) = 0$. ☐

Lemma 694 ***(length of partition).*** *Let $a, b, c \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have*

$$\ell((a,b) \cap (c,\infty)) + \ell((a,b) \cap (c,\infty)^c) = \ell((a,b)). \tag{12.3}$$

Proof. Let $I \stackrel{\text{def.}}{=} (a,b)$, and $E \stackrel{\text{def.}}{=} (c,\infty)$. We have $E^c = (-\infty, c]$.

Case $c \leqslant a$. Then, we have $I \cap E = I$ and $I \cap E^c = \varnothing$. Hence, from Lemma 693 (*length is homogeneous*), we have $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$.

Case $a < c < b$. Then, we have $I \cap E = (a,c)$ and $I \cap E^c = [c,b)$. Hence, from Definition 691 (*length of interval*), and **additive abelian group properties of $\mathbb{R}$**, we have

$$\ell(I \cap E) + \ell(I \cap E^c) = \ell((a,c)) + \ell([c,b)) = (c-a) + (b-c) = b - a = \ell(I).$$

Case $b \leqslant c$. Then, we have $I \cap E = \varnothing$ and $I \cap E^c = I$. Hence, from Lemma 693 (*length is homogeneous*), we have $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$.

Therefore, we always have $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$. ☐

Definition 695 ***(set of countable cover with bounded open intervals).***
Let $A \subset \mathbb{R}$. The *set of countable covers of A with bounded open intervals* is

$$C_A \stackrel{\text{def.}}{=} \left\{ ((a_n,b_n))_{n\in\mathbb{N}} \;\middle|\; (\forall n \in \mathbb{N},\ a_n, b_n \in \mathbb{R} \wedge a_n \leqslant b_n) \quad \wedge \quad A \subset \bigcup_{n\in\mathbb{N}} (a_n, b_n) \right\}. \tag{12.4}$$

Lemma 696 ***(set of countable cover with bounded open intervals is nonempty).***
Let $A \subset \mathbb{R}$. Then, we have $C_A \neq \varnothing$.

Proof. For all $n \in \mathbb{N}$, let $I_n \stackrel{\text{def.}}{=} (-n, n)$. Then, from **the Archimedean property of $\mathbb{R}$**, we have $\mathbb{R} \subset \bigcup_{n\in\mathbb{N}} I_n$. Therefore, from Definition 695 (*set of countable cover with bounded open intervals*), and **properties of inclusion**, we have $(I_n)_{n\in\mathbb{N}} \in C_A$. ☐

Definition 697 ***($\lambda^\star$, Lebesgue measure candidate).***
The *Lebesgue measure candidate* is the function $\lambda^\star : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}_+$ defined by

$$\forall A \subset \mathbb{R}, \quad \lambda^\star(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n\in\mathbb{N}} \ell(I_n) \;\middle|\; (I_n)_{n\in\mathbb{N}} \in C_A \right\}. \tag{12.5}$$

Remark 698. Note that $\lambda^\star$ in the previous definition is not σ-additive on $\mathcal{P}(\mathbb{R})$, e.g. see [28, Ex. 2.28 pp. 86–87]. Thus, it is not a measure on $\mathcal{P}(\mathbb{R})$. In fact, there is no measure on $\mathcal{P}(\mathbb{R})$ that generalizes the length of interval, e.g. see [28, Ex. 2.29 pp. 87–88].

Lemma 699 ***($\lambda^\star$ is nonnegative).*** *Let $A \subset \mathbb{R}$. Then, we have $\lambda^\star(A) \geqslant 0$.*

Proof. Direct consequence of Lemma 692 (*length is nonnegative*), and **monotonicity of infimum**. □

Lemma 700 ***($\lambda^\star$ is homogeneous).*** *We have $\lambda^\star(\varnothing) = 0$.*

Proof. From Lemma 699 (*$\lambda^\star$ is nonnegative*), we have $\lambda(\varnothing) \geqslant 0$.

For all $n \in \mathbb{N}$, let $I_n \stackrel{\text{def.}}{=} (0,0) = \varnothing$. Let $n \in \mathbb{N}$. Then, from Definition 691 (*length of interval*), and **field properties of $\mathbb{R}$**, we have $\ell(I_n) = 0 - 0 = 0$. Thus, we have $\sum_{n\in\mathbb{N}} \ell(I_n) = 0$. Moreover, we have $\varnothing \subset \bigcup_{n\in\mathbb{N}} I_n$. Thus, we have $(I_n)_{n\in\mathbb{N}} \in C_\varnothing$. Hence, from Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), and Definition 9 (*infimum*, lower bound), we have $\lambda^\star(\varnothing) \leqslant 0$.

Therefore $\lambda^\star(\varnothing) = 0$. □

Lemma 701 ***($\lambda^\star$ is monotone).*** *Let $A, B \subset \mathbb{R}$. Assume that $A \subset B$. Then, $\lambda^\star(A) \leqslant \lambda^\star(B)$.*

Proof. Let $(I_n)_{n\in\mathbb{N}} \in C_B$. Then, from Definition 695 (*set of countable cover with bounded open intervals*), and **transitivity of inclusion**, we have $A \subset B \subset \bigcup_{n\in\mathbb{N}} I_n$. Thus, from Definition 695 (*set of countable cover with bounded open intervals*), we have $(I_n)_{n\in\mathbb{N}} \in C_A$. Hence, from Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), and Definition 9 (*infimum*, lower bound, for A), we have $\lambda^\star(A) \leqslant \sum_{n\in\mathbb{N}} \ell(I_n)$.

Therefore, from Definition 9 (*infimum*, greatest lower bound, for B), and Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), we have $\lambda^\star(A) \leqslant \lambda^\star(B)$. □

Lemma 702 ***($\lambda^\star$ is σ-subadditive).*** *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. Then, $\lambda^\star\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_{n\in\mathbb{N}} \lambda^\star(A_n)$.*

Proof. Let $A \stackrel{\text{def.}}{=} \bigcup_{n\in\mathbb{N}} A_n$.

Case $\exists n_0 \in \mathbb{N}$ such that $\lambda^\star(A_{n_0}) = \infty$. Then, from **totally ordered set properties of $\overline{\mathbb{R}}_+$**, we have $\lambda^\star\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \infty = \sum_{n\in\mathbb{N}} \lambda^\star(A_n)$.

Case $\forall n \in \mathbb{N}$, $\lambda^\star(A_n) < \infty$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$. Then, from Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), and Lemma 11 (*finite infimum*), there exists $(I_{n,m})_{m\in\mathbb{N}} \in C_{A_n}$ such that $\sum_{m\in\mathbb{N}} \ell(I_{n,m}) < \lambda^\star(A_n) + \frac{\varepsilon}{2^{n+1}}$. Let $\varphi : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Then, from Definition 695 (*set of countable cover with bounded open intervals*), Lemma 213 (*double countable union*), Lemma 212 (*definition of double countable union*), and **countability of $\mathbb{N}^2$**, we have

$$A = \bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} \left(\bigcup_{m\in\mathbb{N}} I_{n,m} \right) = \bigcup_{n,m\in\mathbb{N}} I_{n,m} = \bigcup_{p\in\mathbb{N}} I_{\varphi(p)}.$$

Then, from Definition 695 (*set of countable cover with bounded open intervals*), $(I_{\varphi(n)})_{n\in\mathbb{N}}$ belongs to C_A, and thus, from Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), Definition 9 (*infimum*, lower bound), Lemma 325 (*definition of double series in $\overline{\mathbb{R}}_+$*), Lemma 326 (*double series in $\overline{\mathbb{R}}_+$*), and **additive properties of $\mathbb{R}_+$**, we have

$$\lambda^\star(A) \leqslant \sum_{p\in\mathbb{N}} \ell(I_{\varphi(p)}) = \sum_{n,m\in\mathbb{N}} \ell(I_{n,m}) = \sum_{n\in\mathbb{N}} \left(\sum_{m\in\mathbb{N}} \ell(I_{n,m}) \right) \leqslant \sum_{n\in\mathbb{N}} \lambda^\star(A_n) + \varepsilon.$$

Hence, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\lambda^\star(A) \leqslant \sum_{n\in\mathbb{N}} \lambda^\star(A_n)$.

Therefore, we always have $\lambda^\star(A) \leqslant \sum_{n\in\mathbb{N}} \lambda^\star(A_n)$. □

Lemma 703 ***($\lambda^\star$ generalizes length of interval).***
Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have $\lambda^\star(\llbracket a, b\rrbracket) = b - a$.

Proof. Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$.

(1). Closed interval. Let $\varepsilon > 0$. Then, we have $[a, b] \subset (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$. Thus, from Definition 695 (*set of countable cover with bounded open intervals*), Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), Definition 9 (*infimum*, lower bound), Lemma 700 (*$\lambda^\star$ is homogeneous*), Definition 691 (*length of interval*), and **ordered field properties of** $\mathbb{R}$, the sequence $((a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}), \varnothing, \ldots)$ belongs to $C_{[a,b]}$, and $\lambda^\star([a, b]) \leqslant \ell(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) = b - a + \varepsilon$. Hence, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\lambda^\star([a, b]) \leqslant b - a$.

Let $((a_n, b_n))_{n\in\mathbb{N}} \in C_{[a,b]}$ (with $\forall n \in \mathbb{N}$, $a_n \leqslant b_n$). Then, from Definition 695 (*set of countable cover with bounded open intervals*), and Lemma 272 (*finite cover of compact interval*), there exists $q \in \mathbb{N}$ and $(i_p)_{p\in[0..q]} \in \mathbb{N}$ pairwise distinct such that $[a, b] \subset \bigcup_{p\in[0..q]}(a_{i_p}, b_{i_p})$ with $a_{i_0} < a$, $b < b_{i_q}$, and for all $p \in [0..q-1]$, $a_{i_{p+1}} < b_{i_p}$. Then, from **ordered field properties of** $\mathbb{R}$, Definition 691 (*length of interval*), and **totally ordered set properties of** $\overline{\mathbb{R}}_+$, we have

$$\begin{aligned} b - a < b_{i_q} - a_{i_0} \leqslant b_{i_q} + \sum_{p\in[0..q-1]} (-a_{i_{p+1}} + b_{i_p}) - a_{i_0} &= \sum_{p\in[0..q]} (b_{i_p} - a_{i_p}) \\ &= \sum_{p\in[0..q]} \ell((a_{i_p}, b_{i_p})) \leqslant \sum_{n\in\mathbb{N}} \ell((a_n, b_n)). \end{aligned}$$

Hence, from Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), and Definition 9 (*infimum*, greatest lower bound), we have $b - a \leqslant \lambda^\star([a, b])$.

Therefore, we have $\lambda^\star([a, b]) = b - a$.

(2). Open interval. Let $\varepsilon > 0$. Assume that $\varepsilon < b - a$. Then, we have

$$\left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right] \subset (a, b) \subset [a, b].$$

Hence, from (1), and Lemma 701 (*$\lambda^\star$ is monotone*), we have $b - a - \varepsilon \leqslant \lambda^\star((a, b)) \leqslant b - a$. Therefore, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\lambda^\star((a, b)) = b - a$.

(3). Left-open right-closed interval. Let $\varepsilon > 0$. Assume that $\varepsilon < b - a$. Then, we have

$$[a + \varepsilon, b] \subset (a, b] \subset [a, b].$$

Hence, from (1), and Lemma 701 (*$\lambda^\star$ is monotone*), we have $b - a - \varepsilon \leqslant \lambda^\star((a, b]) \leqslant b - a$. Therefore, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\lambda^\star((a, b]) = b - a$.

(4). Left-closed right-open interval. Let $\varepsilon > 0$. Assume that $\varepsilon < b - a$. Then, we have

$$(a, b - \varepsilon] \subset [a, b) \subset [a, b].$$

Hence, from (1), and Lemma 701 (*$\lambda^\star$ is monotone*), we have $b - a - \varepsilon \leqslant \lambda^\star([a, b)) \leqslant b - a$. Therefore, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\lambda^\star([a, b)) = b - a$. □

Remark 704.
Note that similar proofs using monotonicity provide for all $a \in \mathbb{R}$, $\lambda^\star((-\infty, a\rrbracket) = \lambda^\star(\llbracket a, \infty)) = \infty$.

Definition 705 ***($\mathcal{L}$, Lebesgue σ-algebra).*** The set $\mathcal{L}$ of subsets of $\mathbb{R}$ defined by

$$\mathcal{L} \overset{\text{def.}}{=} \{E \subset \mathbb{R} \,|\, \forall A \subset \mathbb{R},\ \lambda^\star(A) = \lambda^\star(A \cap E) + \lambda^\star(A \cap E^c)\} \tag{12.6}$$

is called the *Lebesgue σ-algebra*.

Remark 706. The set $\mathcal{L}$ is shown below to be a σ-algebra; hence its name.

Lemma 707 ***(equivalent definition of $\mathcal{L}$).*** *We have*

$$\mathcal{L} = \{E \subset \mathbb{R} \,|\, \forall A \subset \mathbb{R},\ \lambda^\star(A \cap E) + \lambda^\star(A \cap E^c) \leqslant \lambda^\star(A)\}\,. \tag{12.7}$$

Proof. Direct consequence of Lemma 209 (*compatibility of pseudopartition with intersection*, with card(I) equals 2), Lemma 702 (*$\lambda^\star$ is σ-subadditive*), Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, antisymmetry), and Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*). □

Lemma 708 ***($\mathcal{L}$ is closed under complement).*** *$\mathcal{L}$ is closed under complement.*

Proof. Let $E \in \mathcal{L}$. Let $A \subset \mathbb{R}$. Then, from **the double complement law**, Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*), and Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*), we have

$$\lambda^\star(A \cap E^c) + \lambda^\star(A \cap (E^c)^c) = \lambda^\star(A \cap E^c) + \lambda^\star(A \cap E) = \lambda^\star(A \cap E) + \lambda^\star(A \cap E^c) = \lambda^\star(A).$$

Therefore, $E^c \in \mathcal{L}$. □

Lemma 709 ***($\mathcal{L}$ is closed under finite union).*** *$\mathcal{L}$ is closed under finite union.*

Proof. For all $n \in [2..\infty)$, let $P(n)$ be the property: $\forall (E_i)_{i \in [1..n]} \in \mathcal{L}$, $\bigcup_{i \in [1..n]} E_i \in \mathcal{L}$.

Induction: $P(2)$. Let $E_1, E_2 \in \mathcal{L}$. Let $E \stackrel{\text{def.}}{=} E_1 \cup E_2$.
Let $A \subset \mathbb{R}$. Then, from **properties of intersection, union and complement**, we have

$$A \cap E = A \cap (E_1 \uplus (E_1^c \cap E_2)) = (A \cap E_1) \cup (A \cap (E_1^c \cap E_2)).$$

Thus, from Lemma 702 (*$\lambda^\star$ is σ-subadditive*), **associativity of intersection**, and Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*, E_2, then E_1 belong to $\mathcal{L}$), we have

$$\begin{aligned}
\lambda^\star(A \cap E) + \lambda^\star(A \cap E^c) &= \lambda^\star((A \cap E_1) \cup (A \cap (E_1^c \cap E_2))) + \lambda^\star(A \cap (E_1^c \cap E_2^c)) \\
&\leqslant \lambda^\star(A \cap E_1) + \lambda^\star((A \cap E_1^c) \cap E_2) + \lambda^\star((A \cap E_1^c) \cap E_2^c) \\
&= \lambda^\star(A \cap E_1) + \lambda^\star(A \cap E_1^c) \\
&= \lambda^\star(A).
\end{aligned}$$

Hence, from Lemma 707 (*equivalent definition of $\mathcal{L}$*), we have $E \in \mathcal{L}$.

Induction; $P(n)$ implies $P(n+1)$. Let $n \in [2..\infty)$. Assume that $P(n)$ holds.
Let $(E_i)_{i \in [1..n+1]} \in \mathcal{L}$. Let $E \stackrel{\text{def.}}{=} \bigcup_{i \in [1..n]} E_i$. Then, from $P(n)$, we have $E \in \mathcal{L}$, and from **associativity of union**, and $P(2)$, we have $\bigcup_{i \in [1..n+1]} E_i = E \cup E_{n+1} \in \mathcal{L}$.

Therefore, $P(n)$ holds for all $n \in [2..\infty)$. □

Lemma 710 ***($\mathcal{L}$ is closed under finite intersection).*** *$\mathcal{L}$ is closed under finite intersection.*

Proof. Direct consequence of **De Morgan's laws**, Lemma 708 (*$\mathcal{L}$ is closed under complement*), and Lemma 709 (*$\mathcal{L}$ is closed under finite union*). □

Lemma 711 ***($\mathcal{L}$ is set algebra).*** *$\mathcal{L}$ is a set algebra on $\mathbb{R}$.*

Proof. Let $A \subset \mathbb{R}$. Then, from **properties of intersection**, and Lemma 700 (*$\lambda^\star$ is homogeneous*), we have $\lambda^\star(A \cap \mathbb{R}) + \lambda^\star(A \cap \mathbb{R}^c) = \lambda^\star(A) + \lambda^\star(\varnothing) = \lambda^\star(A)$. Hence, $\mathbb{R} \in \mathcal{L}$.

Therefore, from Lemma 708 (*$\mathcal{L}$ is closed under complement*), Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*), Lemma 414 (*closedness under intersection and set difference*, $\mathcal{L}$ is closed under set difference), Lemma 439 (*other equivalent definition of set algebra*), $\mathcal{L}$ is a set algebra on $\mathbb{R}$. □

Lemma 712 ***($\lambda^\star$ is additive on $\mathcal{L}$).*** *Let $n \in [2..\infty)$. Let $(E_i)_{i\in[1..n]} \in \mathcal{L}$. Assume that the E_i's are pairwise disjoint. Then, $\lambda^\star$ is additive on $\mathcal{L}$:*

$$\forall A \subset \mathbb{R}, \quad \lambda^\star\left(A \cap \biguplus_{i\in[1..n]} E_i\right) = \sum_{i\in[1..n]} \lambda^\star(A \cap E_i). \tag{12.8}$$

Proof. For all $n \in [2..\infty)$, let $P(n)$ be the property: for all $(E_i)_{i\in[1..n]} \in \mathcal{L}$,

$$(\forall i,j \in [1..n],\ i \neq j \Rightarrow E_i \cap E_j = \varnothing) \quad \Longrightarrow \quad \lambda^\star\left(A \cap \biguplus_{i\in[1..n]} E_i\right) = \sum_{i\in[1..n]} \lambda^\star(A \cap E_i).$$

Induction: $P(2)$. Let $E_1, E_2 \in \mathcal{L}$. Assume that $E_1 \cap E_2 = \varnothing$. Let $A \subset \mathbb{R}$. Then, from **properties of intersection, union, and complement**, Lemma 209 (*compatibility of pseudopartition with intersection*, with $\mathrm{card}(I) = 2$), and Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*, $E_1 \in \mathcal{L}$), we have

$$\begin{aligned}
\lambda^\star(A \cap (E_1 \uplus E_2)) &= \lambda^\star((A \cap E_1) \uplus (A \cap E_2)) \\
&= \lambda^\star([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1) + \lambda^\star([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1^c) \\
&= \lambda^\star(A \cap E_1) + \lambda^\star(A \cap E_2).
\end{aligned}$$

Induction: $P(n)$ implies $P(n+1)$. Let $n \in [2..\infty)$. Assume that $P(n)$ holds. Let $(E_i)_{i\in[1..n+1]} \in \mathcal{L}$. Assume that for all $i,j \in [1..n+1]$, $i \neq j$ implies $E_i \cap E_j = \varnothing$. Let E be the disjoint union $\biguplus_{i\in[1..n]} E_i$. Then, from **properties of intersection, union, and complement**, $P(2)$, and $P(n)$, we have $E \cap E_{n+1} = \varnothing$, and

$$\lambda^\star(A \cap (E \uplus E_{n+1})) = \lambda^\star(A \cap E) + \lambda^\star(A \cap E_{n+1}) = \sum_{i\in[1..n]} \lambda^\star(A \cap E_i) + \lambda^\star(A \cap E_{n+1}).$$

Therefore, $P(n)$ holds for all $n \in [2..\infty)$. □

Remark 713. The additive property of $\lambda^\star$ *per se* actually corresponds to the case $A = \mathbb{R}$.

Lemma 714 ***($\lambda^\star$ is σ-additive on $\mathcal{L}$).*** *$\lambda^\star$ is σ-additive on $(\mathbb{R}, \mathcal{L})$.*

Proof. Let $(E_n)_{n\in\mathbb{N}} \in \mathcal{L}$. Assume that for all $i,j \in \mathbb{N}$, $i \neq j$ implies $E_i \cap E_j = \varnothing$. Let $E \stackrel{\text{def.}}{=} \bigcup_{n\in\mathbb{N}} E_n$. Let $n \in \mathbb{N}$. Then, from Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*, with $A \stackrel{\text{def.}}{=} \mathbb{R}$), and Lemma 701 (*$\lambda^\star$ is monotone*), we have

$$\sum_{i\in[0..n]} \lambda^\star(E_i) = \lambda^\star\left(\bigcup_{i\in[0..n]} E_i\right) \leqslant \lambda^\star(E).$$

Thus, from **monotonicity of the limit**, we have

$$\sum_{n\in\mathbb{N}} \lambda^\star(E_n) \leqslant \lambda^\star(E).$$

Hence, from Lemma 702 (*$\lambda^\star$ is σ-subadditive*), we have

$$\lambda^\star(E) = \sum_{n\in\mathbb{N}} \lambda^\star(E_n).$$

Therefore, from Definition 608 (*σ-additivity over measurable space*), $\lambda^\star$ is σ-additive over $(\mathbb{R}, \mathcal{L})$. □

Lemma 715 ***(partition of countable union in $\mathcal{L}$).***
Let $(E_n)_{n\in\mathbb{N}} \in \mathcal{L}$. Let $F_0 \overset{\text{def.}}{=} E_0$, and for all $n \in \mathbb{N}$, let $F_{n+1} \overset{\text{def.}}{=} E_{n+1} \setminus \bigcup_{i\in[0..n]} F_i$. Then,

$$\forall n \in \mathbb{N}, \quad F_n \in \mathcal{L}, \tag{12.9}$$

$$\forall m, n \in \mathbb{N}, \quad m \neq n \quad \Longrightarrow \quad F_m \cap F_n = \varnothing, \tag{12.10}$$

$$\forall n \in \mathbb{N}, \quad \bigcup_{i\in[0..n]} E_i = \biguplus_{i\in[0..n]} F_i. \tag{12.11}$$

Proof. Direct consequence of Lemma 711 ($\mathcal{L}$ *is set algebra*), and Lemma 446 (*partition of countable union in set algebra*). □

Lemma 716 ***($\mathcal{L}$ is closed under countable union).*** *$\mathcal{L}$ is closed under countable union.*

Proof. Let $I \subset \mathbb{N}$. Let $(E_i)_{i\in I} \in \mathcal{L}$. Let $E \overset{\text{def.}}{=} \bigcup_{i\in I} E_i$.

Case $\mathbf{card}(I) < \infty$. Then, from Lemma 709 (*$\mathcal{L}$ is closed under finite union*), we have $E \in \mathcal{L}$.

Case $\mathbf{card}(I) = \infty$. Let $\varphi : \mathbb{N} \to I$ be a bijection. Then, from **associativity of union**, we have $E = \bigcup_{n\in\mathbb{N}} E_{\varphi(n)}$. Let $F_0 \overset{\text{def.}}{=} E_{\varphi(0)}$, and for all $n \in \mathbb{N}$, let $F_{n+1} \overset{\text{def.}}{=} E_{\varphi(n+1)} \setminus \bigcup_{i\in[0..n]} F_i$.

Let $n \in \mathbb{N}$. Then, from Lemma 715 (*partition of countable union in $\mathcal{L}$*, $\biguplus_{i\in[0..n]} F_i$ is equal to $\bigcup_{i\in[0..n]} E_{\varphi(i)}$), **properties of union**, and **monotonicity of complement**, we have

$$E^c \subset F^c \quad \text{where } F \overset{\text{def.}}{=} \biguplus_{i\in[0..n]} F_i.$$

Let $A \subset \mathbb{R}$. Then, from Lemma 701 (*$\lambda^\star$ is monotone*, with $A \cap E^c \subset A \cap F^c$), Lemma 715 (*partition of countable union in $\mathcal{L}$*, the F_i's are disjoint in $\mathcal{L}$), Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*), Lemma 709 (*$\mathcal{L}$ is closed under finite union*, $F \in \mathcal{L}$), and Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*, with F), we have

$$\begin{aligned}\sum_{i\in[0..n]} \lambda^\star(A \cap F_i) + \lambda^\star(A \cap E^c) &\leqslant \sum_{i\in[0..n]} \lambda^\star(A \cap F_i) + \lambda^\star(A \cap F^c)\\ &= \lambda^\star(A \cap F) + \lambda^\star(A \cap F^c) = \lambda^\star(A).\end{aligned}$$

Thus, from Lemma 702 (*$\lambda^\star$ is σ-subadditive*), **distributivity of intersection over union**, and **monotonicity of the limit**, we have

$$\lambda^\star\left(A \cap \bigcup_{n\in\mathbb{N}} F_n\right) + \lambda^\star(A \cap E^c) \leqslant \sum_{n\in\mathbb{N}} \lambda^\star(A \cap F_n) + \lambda^\star(A \cap E^c) \leqslant \lambda^\star(A).$$

Hence, since **the limit is a function**, we have $\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_{\varphi(n)} = E$, and

$$\lambda^\star(A \cap E) + \lambda^\star(A \cap E^c) \leqslant \lambda^\star(A).$$

Therefore, from Lemma 707 (*equivalent definition of $\mathcal{L}$*), we have $E \in \mathcal{L}$. □

Lemma 717 ***(rays are Lebesgue-measurable).*** *Let $a \in \mathbb{R}$. Then, we have*

$$(a, \infty),\ (-\infty, a],\ (-\infty, a),\ [a, \infty),\ \in \mathcal{L}. \tag{12.12}$$

Proof. **(1). $(a, \infty) \in \mathcal{L}$.**
Let $A \subset \mathbb{R}$. Let $\alpha \overset{\text{def.}}{=} \lambda^\star(A \cap (a, \infty)) + \lambda^\star(A \cap (-\infty, a])$. Let us show that $\alpha \leqslant \lambda^\star(A)$.

Case $\lambda^\star(A) = \infty$. Trivial. **Case $\lambda^\star(A)$ finite.** Let $\varepsilon > 0$. From Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*), and Lemma 11 (*finite infimum*), let $(I_n)_{n\in\mathbb{N}} \in C_A$ such that

$$\lambda^\star(A) \leqslant \sum_{n\in\mathbb{N}} \ell(I_n) \leqslant \lambda^\star(A) + \varepsilon.$$

For all $n \in \mathbb{N}$, let $I'_n \stackrel{\text{def.}}{=} I_n \cap (a, \infty)$ and $I''_n \stackrel{\text{def.}}{=} I_n \cap (-\infty, a]$. Let $n \in \mathbb{N}$. From Lemma 246 (*intervals are closed under finite intersection*), I'_n and I''_n are intervals (possibly empty). Then, from Lemma 703 (*$\lambda^\star$ generalizes length of interval*), and since I'_n and I''_n are contiguous, we have $\lambda^\star(I'_n) + \lambda^\star(I''_n) = \ell(I'_n) + \ell(I''_n) = \ell(I_n)$. Hence, from Lemma 701 (*$\lambda^\star$ is monotone*), and Lemma 702 (*$\lambda^\star$ is σ-subadditive*), we have

$$\alpha \leqslant \lambda^\star\left(\bigcup_{n\in\mathbb{N}} I'_n\right) + \lambda^\star\left(\bigcup_{n\in\mathbb{N}} I''_n\right) \leqslant \sum_{n\in\mathbb{N}} (\lambda^\star(I'_n) + \lambda^\star(I''_n)) = \sum_{n\in\mathbb{N}} \ell(I_n) \leqslant \lambda^\star(A) + \varepsilon.$$

Therefore, from **monotonicity of the limit (when $\varepsilon \to 0^+$)**, we have $\alpha \leqslant \lambda^\star(A)$, and from Lemma 707 (*equivalent definition of $\mathcal{L}$*), we have $(a, \infty) \in \mathcal{L}$.

(2). $(-\infty, a] \in \mathcal{L}$.
Direct consequence of Lemma 708 (*$\mathcal{L}$ is closed under complement*), and (1).

(3). $(-\infty, a) \in \mathcal{L}$.
Direct consequence of Lemma 716 (*$\mathcal{L}$ is closed under countable union*), and (2) with

$$(-\infty, a) = \bigcup_{n\in\mathbb{N}} \left(-\infty, a - \frac{1}{n+1}\right].$$

(4). $[a, \infty) \in \mathcal{L}$.
Direct consequence of Lemma 708 (*$\mathcal{L}$ is closed under complement*), and (3). □

Lemma 718 ***(intervals are Lebesgue-measurable).***
Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have

$$(a, b),\ [a, b],\ [a, b),\ (a, b]\ \in \mathcal{L}. \tag{12.13}$$

Proof. Direct consequence of Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*), and Lemma 717 (*rays are Lebesgue-measurable*) with

$$(a, b) = (-\infty, b) \cap (a, \infty), \quad [a, b] = (-\infty, b] \cap [a, \infty),$$
$$[a, b) = (-\infty, b) \cap [a, \infty), \quad (a, b] = (-\infty, b] \cap (a, \infty).$$

□

Lemma 719 ***($\mathcal{L}$ is σ-algebra).*** *$(\mathbb{R}, \mathcal{L})$ is a measurable space.*

Proof. Direct consequence of Lemma 711 (*$\mathcal{L}$ is set algebra*), Definition 437 (*set algebra*), Lemma 716 (*$\mathcal{L}$ is closed under countable union*), and Definition 474 (*σ-algebra*). □

Lemma 720 ***($\lambda^\star$ is measure on $\mathcal{L}$).*** *$(\mathbb{R}, \mathcal{L}, \lambda^\star_{|\mathcal{L}})$ is a measure space.*

Proof. Direct consequence of Lemma 719 (*$\mathcal{L}$ is σ-algebra*), Lemma 699 (*$\lambda^\star$ is nonnegative*), Lemma 700 (*$\lambda^\star$ is homogeneous*), Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*), and Definition 611 (*measure*). □

Lemma 721 ***($\mathcal{B}(\mathbb{R})$ is sub-σ-algebra of $\mathcal{L}$).*** *We have $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$.*

Proof. Direct consequence of Lemma 484 (*σ-algebra generation is monotone*), Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*), and Lemma 718 (*intervals are Lebesgue-measurable*). □

Lemma 722 ***($\lambda^\star$ is measure on $\mathcal{B}(\mathbb{R})$).*** *$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^\star_{|\mathcal{B}(\mathbb{R})})$ is a measure space.*

Proof. Direct consequence of Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*), and Lemma 721 (*$\mathcal{B}(\mathbb{R})$ is sub-σ-algebra of $\mathcal{L}$*). □

Remark 723. See the sketch of next proof in Section 5.6.

Theorem 724 ***(Carathéodory, Lebesgue measure on $\mathbb{R}$).*** *There exists a unique measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that generalizes the length of bounded open intervals.*

This measure is denoted $\lambda \overset{\text{def.}}{=} \lambda^\star_{|\mathcal{B}(\mathbb{R})}$; it is called the (Borel–)Lebesgue measure on Borel subsets (of $\mathbb{R}$).

Proof. **Existence.** Direct consequence of Lemma 722 (*$\lambda^\star$ is measure on $\mathcal{B}(\mathbb{R})$*), and Lemma 703 (*$\lambda^\star$ generalizes length of interval*, for open intervals).

Uniqueness.
Let $G \overset{\text{def.}}{=} \{[a,b)\}_{a<b}$. Then, we have $G \subset \mathcal{P}(\mathbb{R})$ and $G \neq \varnothing$. Hence, from Lemma 246 (*intervals are closed under finite intersection*), and Definition 428 (*π-system*), G is a π-system.
From Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*, $\{(a,b)\}_{a<b}$ and $\{[a,b)\}_{a<b}$ generate $\mathcal{B}(\mathbb{R})$), Lemma 483 (*generated σ-algebra is minimum*, generators belong to $\mathcal{B}(\mathbb{R})$), we have

$$(\forall a,b \in \mathbb{R}, \quad a < b \implies (a,b), [a,b) \in \mathcal{B}(\mathbb{R})) \quad \text{and} \quad \Sigma_{\mathbb{R}}(G) = \mathcal{B}(\mathbb{R}).$$

For all $n \in \mathbb{N}$, let $I_n \overset{\text{def.}}{=} [n, n+1)$. Then, $(I_n)_{n\in\mathbb{N}}$ is obviously pairwise disjoint, and from **the Archimedean property of $\mathbb{R}$**, we have $\mathbb{R} = \biguplus_{n\in\mathbb{N}} I_n$. Let $n \in \mathbb{N}$. Then, $I_n \in G$, and from Lemma 703 (*$\lambda^\star$ generalizes length of interval*), we have $\lambda^\star(I_n) = 1 < \infty$.

Let μ be a measure on $\mathcal{B}(\mathbb{R})$ such that, for all $a, b \in \mathbb{R}$, $a < b$ implies $\mu((a,b)) = b - a$.
Let $a, b \in \mathbb{R}$. Assume that $a < b$. For all $p \in \mathbb{N}$, let $A_p \overset{\text{def.}}{=} \left(a - \frac{1}{p+1}, b\right)$. Then, from **the nonincreasing property of $\left(p \mapsto \frac{1}{p+1}\right)$ in $\mathbb{R}_+$ and its limit 0 when $p \to \infty$**, **the definition of inclusion**, **additive group properties of $\mathbb{R}$**, **linearity of the limit**, and Lemma 619 (*measure is continuous from above*, with μ_1 and $(A_p)_{p\in\mathbb{N}}$), the sequence $(A_p)_{p\in\mathbb{N}}$ is nonincreasing, for all $p \in \mathbb{N}$, we have $\mu_1(A_p) = b - a + \frac{1}{p+1} < \infty$, and

$$\mu_1([a,b)) = \mu_1\left(\bigcap_{p\in\mathbb{N}} A_p\right) = \inf_{p\in\mathbb{N}} \mu_1(A_p) = \lim_{p\to\infty} \left(b - a + \frac{1}{p+1}\right) = b - a.$$

Hence, from Lemma 703 (*$\lambda^\star$ generalizes length of interval*), μ and $\lambda^\star_{|\mathcal{B}(\mathbb{R})}$ coincide on G.

Therefore, from Lemma 668 (*uniqueness of measures extended from a π-system*, with $X \overset{\text{def.}}{=} \mathbb{R}$, $\Sigma \overset{\text{def.}}{=} \mathcal{B}(\mathbb{R})$, $\mu_1 \overset{\text{def.}}{=} \lambda^\star$, $\mu_2 = \mu$, and $X_n \overset{\text{def.}}{=} I_n$), we have, for all $A \in \mathcal{B}(\mathbb{R})$, $\mu(A) = \lambda^\star_{|\mathcal{B}(\mathbb{R})}(A)$. □

Remark 725. Note that $\mathbf{N}(\mathbb{R}, \mathcal{L}, \lambda^\star) = \mathbf{N}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \subset \mathcal{L}$, e.g. see [28, Ex. 2.33 pp. 92–94].

Lemma 726 ***(Lebesgue measure generalizes length of interval).*** *Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have $\lambda([\![a,b]\!]) = b - a$.*

Proof. Direct consequence of Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*), and Lemma 703 (*$\lambda^\star$ generalizes length of interval*). □

Remark 727. Note that similarly to $\lambda^\star$, we can prove for all $a \in \mathbb{R}$, $\lambda((-\infty, a]\!]) = \lambda([\![a, \infty)) = \infty$.

Lemma 728 ***(Lebesgue measure is σ-finite).*** *The measure space* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ *is* σ*-finite.*

Proof. Direct consequence of Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$), Lemma 703 ($\lambda^\star$ *generalizes length of interval*, with $b = -a \in \mathbb{N}$), **the Archimedean property of** $\mathbb{R}$, and Definition 624 (σ*-finite measure*, with $A_n \stackrel{\text{def.}}{=} (-n, n)$). □

Lemma 729 ***(Lebesgue measure is diffuse).*** *The measure space* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ *is diffuse.*

Proof. Direct consequence of Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$), Lemma 703 ($\lambda^\star$ *generalizes length of interval*, with $b = a$), and Definition 626 (*diffuse measure*, $\{a\} = [a, a]$). □

Chapter 13

Integration of nonnegative functions

Contents

Remark 730.
The first three sections of this chapter follow steps 1 to 3 of the Lebesgue scheme (see Section 4.1).

13.1 Integration of indicator functions

Remark 731. In this section, functions take their values in $\{0, 1\}$, and the expressions involving integrals are taken in $\overline{\mathbb{R}}_+$.

Definition 732 ***($\mathcal{IF}$, set of measurable indicator functions).***
Let (X, Σ) be a measurable space. The *set of measurable indicator functions* is denoted $\mathcal{IF}(X, \Sigma)$ (or simply $\mathcal{IF}$); it is defined by $\mathcal{IF}(X, \Sigma) \stackrel{\text{def.}}{=} \{\mathbb{1}_A \mid A \in \Sigma\}$.

Lemma 733 ***(indicator and support are each other inverse).***
Let (X, Σ) be a measurable space. Let $A \in \Sigma$, let $f \in \mathcal{IF}$. Then, we have

$$\mathbb{1}_A \in \mathcal{IF}, \quad \{f \neq 0\} \in \Sigma, \quad \{\mathbb{1}_A \neq 0\} = A \quad \text{and} \quad \mathbb{1}_{\{f \neq 0\}} = f. \tag{13.1}$$

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), **the definition of the indicator function**, and **the definition of inverse image**. □

Lemma 734 ***($\mathcal{IF}$ is measurable).***
Let (X, Σ) be a measurable space. Then, we have $\mathcal{IF} \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$.

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), Lemma 569 (*measurability of indicator function*), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$), **non-negativeness of indicator function**, and Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*). ☐

Lemma 735 ***($\mathcal{IF}$ is σ-additive).*** *Let (X, Σ) be a measurable space. Let $I \subset \mathbb{N}$. Let $(A_i)_{i \in I} \in \Sigma$. Assume that the A_i's are pairwise disjoint. Then, we have $\sum_{i \in I} \mathbb{1}_{A_i} = \mathbb{1}_{\biguplus_{i \in I} A_i} \in \mathcal{IF}$.*

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), **the formula for the indicator of disjoint union (sum)**, Definition 516 (*measurable space*, Σ is a σ-algebra), and Definition 474 (*σ-algebra*, closedness under countable union). ☐

Lemma 736 ***($\mathcal{IF}$ is closed under multiplication).***
Let (X, Σ) be a measurable space. Let $A, B \in \Sigma$. Then, we have $\mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B} \in \mathcal{IF}$.

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), **the formula for the indicator of intersection (product)**, and Lemma 475 (*equivalent definition of σ-algebra*, closedness under intersection). ☐

Remark 737.
We recall the notation $\mathbb{1}_U^V$ to denote the indicator function of U defined on V (when $U \subset V$).

Lemma 738 ***($\mathcal{IF}$ is closed under extension by zero).***
Let (X, Σ) be a measurable space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$ and $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$ and $f_{|A} \in \mathcal{IF}(A, \Sigma \overline{\cap} A)$. Then, we have $\hat{f}\,\mathbb{1}_A \in \mathcal{IF}(X, \Sigma)$.

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*, $f_{|A} = \mathbb{1}_B^A$ with $B \in \Sigma$ such that $B \subset A$), and Lemma 218 (*restriction is masking*, $\hat{f}\,\mathbb{1}_A = \mathbb{1}_B^X$). ☐

Lemma 739 ***($\mathcal{IF}$ is closed under restriction).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{IF}(X, \Sigma)$. Let $A \in \Sigma$. Then, we have $f_{|A} \in \mathcal{IF}(A, \Sigma \overline{\cap} A)$.*

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*, $f = \mathbb{1}_B^X$ with $B \in \Sigma$), **the rule for the restriction of an indicator function ($f_{|A} = \mathbb{1}_{B \cap A}^A$)**, and Lemma 533 (*measurability of measurable subspace*, $B \cap A \in \Sigma \overline{\cap} A$). ☐

Definition 740 ***(integral in $\mathcal{IF}$).*** Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{IF}$. The *integral of f (for the measure μ)* is denoted $\int f\,d\mu$; it is defined by

$$\int f\,d\mu \stackrel{\text{def.}}{=} \mu(\{f \neq 0\}) \quad \in \overline{\mathbb{R}}_+. \tag{13.2}$$

Lemma 741 ***(equivalent definition of integral in $\mathcal{IF}$).***
Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Then, we have $\int \mathbb{1}_A\,d\mu = \mu(A)$.

Proof. Direct consequence of Lemma 733 (*indicator and support are each other inverse*, $\mathbb{1}_A \in \mathcal{IF}$ and $\{\mathbb{1}_A \neq 0\} = A \in \Sigma$), and Definition 740 (*integral in $\mathcal{IF}$*). ☐

Lemma 742 ***(integral in $\mathcal{IF}$ is additive).*** *Let (X, Σ, μ) be a measure space. Let $n \in \mathbb{N}$. Let $(A_i)_{i \in [0..n]} \in \Sigma$. Assume that the A_i's are pairwise disjoint. Then, we have*

$$\int \sum_{i \in [0..n]} \mathbb{1}_{A_i}\,d\mu = \int \mathbb{1}_{\biguplus_{i \in [0..n]} A_i}\,d\mu = \sum_{i \in [0..n]} \int \mathbb{1}_{A_i}\,d\mu. \tag{13.3}$$

Proof. Direct consequence of Lemma 735 (*$\mathcal{IF}$ is σ-additive*), Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*), Lemma 621 (*equivalent definition of measure*, additivity), and Definition 607 (*additivity over measurable space*). □

Lemma 743 *(integral in $\mathcal{IF}$ over subset).* *Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, we have $f_{|A} \in \mathcal{IF}(A, \Sigma \overline{\cap} A)$ iff $\hat{f}\,\mathbb{1}_A \in \mathcal{IF}(X, \Sigma)$.*

If so, there exists $B \in \Sigma$ such that $B \subset A$, $f_{|A} = \mathbb{1}^A_B$, $\hat{f}\,\mathbb{1}_A = \mathbb{1}^X_B$, and we have

$$\int f_{|A}\, d\mu_A = \int \hat{f}\,\mathbb{1}_A\, d\mu. \tag{13.4}$$

This integral is denoted $\int_A f\, d\mu$; it is called integral of f over A.

Proof. **Equivalence.** Direct consequence of Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*), and Lemma 739 (*$\mathcal{IF}$ is closed under restriction*).

Identity. Direct consequence of Lemma 218 (*restriction is masking*), Lemma 733 (*indicator and support are each other inverse*, $f_{|A} = \mathbb{1}^A_B$ and $\hat{f}\,\mathbb{1}_A = \mathbb{1}^X_B$ with $B \in \Sigma$ such that $B \subset A$), Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*, with $\mu \stackrel{\text{def.}}{=} \mu_A$, then $A \stackrel{\text{def.}}{=} B$), and Lemma 628 (*trace measure*). □

Lemma 744 *(integral in $\mathcal{IF}$ over subset is additive).* *Let (X, Σ, μ) be a measure space. Let $n \in \mathbb{N}$. Let $A, (A_i)_{i\in[0..n]} \in \Sigma$. Assume that $(A_i)_{i\in[0..n]}$ is a pseudopartition of A. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, $\hat{f}\,\mathbb{1}_A \in \mathcal{IF}$ iff for all $i \in [0..n]$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{IF}$. If so, we have*

$$\int_A f\, d\mu = \sum_{i\in[0..n]} \int_{A_i} f\, d\mu. \tag{13.5}$$

Proof. **"Left" implies "right".** Direct consequence of Lemma 733 (*indicator and support are each other inverse*, $\hat{f}\,\mathbb{1}_A = \mathbb{1}_B$ with $B \in \Sigma$ such that $B \subset A$), and Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*, $\hat{f}\,\mathbb{1}_{A_i} = \mathbb{1}_{B\cap A_i}$).

"Right" implies "left". Direct consequence of Lemma 733 (*indicator and support are each other inverse*, $\hat{f}\,\mathbb{1}_{A_i} = \mathbb{1}_{B_i}$ with $B_i \in \Sigma$ such that $B_i \subset A_i$), **monotonicity of intersection**, Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $B \stackrel{\text{def.}}{=} \biguplus_{i\in[0..n]} B_i \in \Sigma$), Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I \stackrel{\text{def.}}{=} [0..n]$), and **left distributivity of multiplication over addition in $\mathbb{R}$ ($\hat{f}\,\mathbb{1}_A = \mathbb{1}_B$)**.

Therefore, we have the equivalence.

Identity. Direct consequence of Lemma 743 (*integral in $\mathcal{IF}$ over subset*, with A and the A_i's), Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I \stackrel{\text{def.}}{=} [0..n]$), **left distributivity of multiplication over addition in $\mathbb{R}$**, and Lemma 742 (*integral in $\mathcal{IF}$ is additive*). □

Remark 745.
In the next lemma, when Y is uncountable, the sum (of nonnegative values) is understood as the supremum for all finite subsets (which is also correct of course in the countable case),

$$\sum_{y\in Y} f(y) \stackrel{\text{def.}}{=} \sup_{\substack{Z\subset Y \\ \text{card}(Z)<\infty}} \sum_{z\in Z} f(z). \tag{13.6}$$

Lemma 746 ***(integral in $\mathcal{IF}$ for counting measure).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{IF}$. Then, we have

$$\int f \, d\delta_Y = \sum_{y \in Y} f(y). \tag{13.7}$$

Proof. Direct consequence of Lemma 733 (*indicator and support are each other inverse*), Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*, with $A = \{f \neq 0\} \in \Sigma$), Lemma 671 (*counting measure*), **the definition of the cardinality**, and **the definition of the indicator function**. □

13.2 Integration of nonnegative simple functions

13.2.1 Simple function

Remark 747. In this section, the functions take their values in $\mathbb{R}$.

Definition 748 ***($\mathcal{SF}$, vector space of simple functions).***
Let (X, Σ) be a measurable space. The vector subspace of finite linear combinations of indicator functions of measurable subsets is called *vector space of simple functions*; it is denoted $\mathcal{SF}(X, \Sigma)$ (or simply $\mathcal{SF}$); it is defined by $\mathcal{SF}(X, \Sigma) \stackrel{\text{def.}}{=} \operatorname{span}(\mathcal{IF}(X, \Sigma))$.

Lemma 749 ***($\mathcal{SF}$ simple representation).***
Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$. Then, we have $f \in \mathcal{SF}$ iff there exists $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}$ and $(A_i)_{i\in[0..n]} \in \Sigma$ such that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$.
If so, for all $x \in X \setminus \bigcup_{i\in[0..n]} A_i$, we have $f(x) = 0$.

Proof. **Equivalence.** Direct consequence of Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), and Lemma 733 (*indicator and support are each other inverse*).

Property.
Direct consequence of **the definition of the indicator function**, and **field properties of $\mathbb{R}$ (0 is absorbing element for multiplication and identity element for addition)**. □

Remark 750. Note that in a simple representation, the values a_i's may not be unique and may not be values of the function, and the supports A_i's may be empty, may not be related to preimages of a_i's, may overlap and thus may not form a partition of X.

Remark 751. We recall that $f^{-1}(y)$ denotes the subset $f^{-1}(\{y\})$.

Lemma 752 ***($\mathcal{SF}$ canonical representation).***
Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$.
Then, we have $f \in \mathcal{SF}$ iff $f(X)$ is finite, and for all $y \in f(X)$, $f^{-1}(y)$ belongs to Σ.
If so, we have the following canonical representation*:*

$$f = \sum_{y\in f(X)} y \, \mathbb{1}_{f^{-1}(y)}, \tag{13.8}$$

i.e. there exists unique $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}$, and $(A_i)_{i\in[0..n]} \in \Sigma$, such that

$$\forall i \in [0..n-1], \quad a_i < a_{i+1}, \tag{13.9}$$

$$\forall i \in [0..n], \quad A_i = f^{-1}(a_i) \neq \varnothing, \tag{13.10}$$

$$\forall p, q \in [0..n], \quad p \neq q \implies A_p \cap A_q = \varnothing, \tag{13.11}$$

$$X = \biguplus_{i\in[0..n]} A_i, \tag{13.12}$$

$$f(X) = \{a_i \mid i \in [0..n]\}, \tag{13.13}$$

$$f = \sum_{i\in[0..n]} a_i \, \mathbb{1}_{A_i}. \tag{13.14}$$

Proof. **"Left" implies "right".** Assume first that $f \in \mathcal{SF}$. Then, from Lemma 749 (*$\mathcal{SF}$ simple representation*), there exists $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}$, and $(A_i)_{i\in[0..n]} \in \Sigma$, such that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$. Thus, from **the definition of the indicator function**, f can only take the values $\sum_{i\in[0..n]} a_i \delta_i$ where $(\delta_i)_{i\in[0..n]} \in \{0, 1\}$. Hence, the cardinality of $f(X)$ is at most 2^{n+1}.

Let $(\delta_i)_{i\in[0..n]} \in \{0,1\}$. Let $y \stackrel{\text{def.}}{=} \sum_{i\in[0..n]} a_i\delta_i \in f(X)$. Let $I_y \subset [0..n]$ such that $i \in I_y$ iff $\delta_i = 1$. Then, we have[1] $y = \sum_{i\in I_y} a_i$. Unfortunately, several partial sums of a_i's may lead to the same value y. Thus, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection and union), we have

$$f^{-1}(y) = \bigcup_{\substack{I_y\subset\mathcal{P}([0..n])\\ \sum_{i\in I_y} a_i=y}} \left(\bigcap_{i\in I_y} A_i\right) \in \Sigma.$$

From **properties of inverse image**, the collection $(f^{-1}(y))_{y\in f(X)}$ makes a finite partition of the whole set X. Let $x \in X$. Then, there exists a unique $y \in \mathbb{R}$ such that $x \in f^{-1}(y)$. Thus, from **field properties of** $\mathbb{R}$, we have $f(x) = y = y \times 1 = y\,\mathbb{1}_{f^{-1}(y)}(x)$. Hence, Equation (13.8) holds. And Equation (13.14) is equivalent, up to a nondecreasing reordering of the $y \in f(X)$.

"Right" implies "left". Let $n \stackrel{\text{def.}}{=} \mathrm{card}(f(X)) - 1 \in \mathbb{N}$. Let $(a_i)_{i\in[0..n]} \in \mathbb{R}$ be the $n+1$ distinct values taken by the function f, i.e. such that $f(X) = \{a_i \,|\, i \in [0..n]\}$. For all $i \in [0..n]$, let

$$A_i \stackrel{\text{def.}}{=} f^{-1}(a_i) \in \Sigma.$$

Then, from **the definition and properties of image and inverse image**, and **the definition of partition**, the n subsets $(A_i)_{i\in[0..n]}$ constitutes a partition of X. Thus, from **the definition of the indicator function**, we have $f = \sum_{i\in[0..n]} a_i\,\mathbb{1}_{A_i}$. Hence, from Lemma 749 (*$\mathcal{SF}$ simple representation*), we have $f \in \mathcal{SF}$.

Uniqueness. Direct consequence of (13.9)–(13.14) since $\mathrm{card}(f(X)) = n+1$ (i.e. n is unique), $f(X) = \{a_i \,|\, i \in [0..n]\}$ (i.e. the a_i's are unique), and $A_i = f^{-1}(a_i)$ (i.e. the A_i's are unique).

Therefore, we have the equivalence, and the representation is unique. □

Remark 753. To sum up, in a canonical representation, the values a_i's are unique and are the values of the function, and the supports A_i's are the nonempty preimages of the a_i's and form a partition of X. Hence, the canonical representation of the zero function is $0\,\mathbb{1}_X$. Note that Equations (13.8) and (13.14) are identical, up to an nondecreasing reordering of the $y \in f(X)$.

Some authors exclude $y = 0$ from the sum over $f(X)$. In this case the partition property expressed in (13.10)–(13.13) weakens into the pairwise disjunction property (13.11) and inclusions instead of equalities. Then, the zero function must be treated differently as its canonical representation becomes $\mathbb{1}_\varnothing$.

Lemma 754 ***($\mathcal{SF}$ disjoint representation).***
Let (X,Σ) be a measurable space. Let $f : X \to \mathbb{R}$.
Then, $f \in \mathcal{SF}$ iff there exists $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}$, and $(A_i)_{i\in[0..n]} \in \Sigma$, such that

$$\forall i \in [0..n],\quad A_i \subset f^{-1}(a_i), \tag{13.15}$$

$$\forall p,q \in [0..n],\quad p \neq q \implies A_p \cap A_q = \varnothing, \tag{13.16}$$

$$X = \biguplus_{i\in[0..n]} A_i, \tag{13.17}$$

$$f = \sum_{i\in[0..n]} a_i\,\mathbb{1}_{A_i}. \tag{13.18}$$

If so, it is called a disjoint representation (of f).

Proof. Direct consequence of Lemma 752 (*$\mathcal{SF}$ canonical representation*, "left" implies "right"), and Lemma 749 (*$\mathcal{SF}$ simple representation*, "right" implies "left"). □

[1] For instance, the case $I_y = \varnothing$ may correspond to $y = 0$ when $0 \notin \{a_i \,|\, i \in [0..n]\}$.

Remark 755. In a disjoint representation, the values a_i's may not be unique ($a_i = a_j$ with $i \neq j$ is possible) and may not be the values of the function ($A_i = \varnothing$ is possible). More precisely, the supports A_i's may be empty but are subsets of the associated preimage of a_i (thus the values that are not taken by the function are associated with an empty support). They form a pseudopartition of X (see Definition 207).

Note that the nonempty A_i's form a subpartition of $(f^{-1}(y))_{y\in f(X)}$ (see next lemma).

Lemma 756 ***($\mathcal{SF}$ disjoint representation is subpartition of canonical representation).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{SF}$. Let $n, m \in \mathbb{N}$, $(a_i)_{i\in[0..m]}, (b_j)_{j\in[0..m]} \in \mathbb{R}$, and $(A_i)_{i\in[0..m]}, (B_j)_{j\in[0..m]} \in \Sigma$. Assume that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$ is a disjoint representation, and that $f = \sum_{j\in[0..m]} b_j \mathbb{1}_{B_j}$ is the canonical representation.*

For all $j \in [0..m]$, let $I'_j \overset{\text{def.}}{=} \{i \in [0..n] \,|\, \varnothing \neq A_i \subset B_j\}$. Let $I' \overset{\text{def.}}{=} \{i \in [0..n] \,|\, A_i \neq \varnothing\}$. Then, we have

$$\forall j \in [0..m], \quad (\forall i \in I'_j,\ a_i = b_j) \quad \wedge \quad B_j = \biguplus_{i\in I'_j} A_i, \tag{13.19}$$

$$(\forall p, q \in [0..m],\ p \neq q \ \Rightarrow\ I'_p \cap I'_q = \varnothing) \quad \wedge \quad \biguplus_{j\in[0..m]} I'_j = I'. \tag{13.20}$$

Proof. Let $i \in [0..n]$ and $j \in [0..m]$. Assume that $A_i \neq \varnothing$.
(0a) $\boldsymbol{A_i \subset B_j \Rightarrow a_i = b_j}$**.** Let $x \in A_i \cap B_j$. Then, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $f(x) = a_i$), and Lemma 752 (*$\mathcal{SF}$ canonical representation*, $f(x) = b_j$), we have $a_i = b_j$.
(0b) $\boldsymbol{a_i = b_j \Rightarrow A_i \subset B_j}$**.** Assume that $a_i = b_j$. Then, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*), and Lemma 752 (*$\mathcal{SF}$ canonical representation*), we have $A_i \subset f^{-1}(a_i) = f^{-1}(b_j) = B_j$.

(1). Let $j \in [0..m]$.
From Lemma 754 (*$\mathcal{SF}$ disjoint representation*), the A_i's are pairwise disjoint. Let $A'_j \overset{\text{def.}}{=} \biguplus_{i\in I'_j} A_i$.
Let $x \in B_j$. Then, from Lemma 752 (*$\mathcal{SF}$ canonical representation*, $f(x) = b_j$), and Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $X = \biguplus_{i\in[0..n]} A_i$ and $A_i \subset f^{-1}(a_i)$), there exists $i \in [0..n]$ such that $x \in A_i$, i.e. $A_i \neq \varnothing$ and $a_i = f(x) = b_j$. Thus, from (0b), we have $i \in I'_j$, and $B_j \subset A'_j$. Hence, since the other inclusion is obvious, we have equality, and from (0a), property (13.19) holds.

(2a). Let $p, q \in [0..m]$. Assume that $p \neq q$.
Let $i \in I'_p \cap I'_q$. Then, from Lemma 752 (*$\mathcal{SF}$ canonical representation*, pairwise disjunction), we have $\varnothing \neq A_i \subset B_p \cap B_q = \varnothing$, which is impossible. Hence, we have $I'_p \cap I'_q = \varnothing$.

(2b). Let $i \in I'$, i.e. $A_i \neq \varnothing$.
Then, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $a_i \in f(X)$), and Lemma 752 (*$\mathcal{SF}$ canonical representation*, $f(X) = \{b_j \,|\, j \in [0..m]\}$), there exists $j \in [0..m]$ such that $a_i = b_j$. Thus, from (0b), we have $i \in I'_j$, and $I' \subset \biguplus_{j\in[0..m]} I'_j$. Hence, since the other inclusion is obvious, we have equality, and from (2a), property (13.20) holds.

Therefore, both properties hold. □

Lemma 757 *($\mathcal{SF}$ is algebra over $\mathbb{R}$).*
Let (X,Σ) be a measurable space. Then, $\mathcal{SF}$ is a subalgebra of $\mathbb{R}^X$.
Let $f,g\in\mathcal{SF}$, $n,m\in\mathbb{N}$, $(a_i)_{i\in[0..n]},(b_j)_{j\in[0..m]}\in\mathbb{R}$, and $(A_i)_{i\in[0..n]},(B_j)_{j\in[0..m]}\in\Sigma$ such that $f=\sum_{i\in[0..n]}a_i\,\mathbb{1}_{A_i}$ and $g=\sum_{j\in[0..m]}b_j\,\mathbb{1}_{B_j}$.
Let $N\stackrel{\text{def.}}{=}nm+n+m$. Let $\varphi:[0..N]\to[0..n]\times[0..m]$ be a bijection.
For all $(i,j)\in[0..n]\times[0..m]$, let $c^+_{i,j}\stackrel{\text{def.}}{=}a_i+b_j$, $c^*_{i,j}\stackrel{\text{def.}}{=}a_ib_j$, and $C_{i,j}\stackrel{\text{def.}}{=}A_i\cap B_j$.
Then, on the one hand, if both representations are disjoint, we have

$$f+g=\sum_{k\in[0..N]}c^+_{\varphi(k)}\,\mathbb{1}_{C_{\varphi(k)}},\tag{13.21}$$

and it is also a disjoint representation. On the other hand, we always have

$$fg=\sum_{k\in[0..N]}c^*_{\varphi(k)}\,\mathbb{1}_{C_{\varphi(k)}},\tag{13.22}$$

and it is also a disjoint representation when those of f and g are.

Proof. Since $N+1=(n+1)(m+1)$, such a bijection φ exists.

(1). From Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), **the definition of the linear span**, and Definition 77 (*subspace*), $\mathcal{SF}$ is a vector subspace of $\mathbb{R}^X$. From **field properties of $\mathbb{R}$**, Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under intersection), we have Equation (13.22) where for all $i\in[0..n]$, $j\in[0..m]$, $A_i\cap B_j\in\Sigma$. Hence, from Lemma 749 (*$\mathcal{SF}$ simple representation*), we have $fg\in\mathcal{SF}$. Therefore, from Lemma 235 (*vector subspace and closed under multiplication is subalgebra*), $\mathcal{SF}$ is a subalgebra of $\mathbb{R}^X$.

Assume now that both representations of f and g are disjoint.

(2a). Then, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*), **compatibility of intersection with pairwise disjunction**, **idempotent law for intersection**, and **left and right distributivity of intersection over union**, the $A_i\cap B_j$'s are pairwise disjoint, and

$$X=\biguplus_{i\in[0..n]}A_i\cap\biguplus_{j\in[0..m]}B_j=\biguplus_{k\in[0..N]}C_{\varphi(k)}.$$

Hence, the $C_{\varphi(k)}$'s form a pseudopartition of X (there may be empty parts in it, see Remark 755).

(2b). Let $i\in[0..n]$. Then, from **field properties of $\mathbb{R}$**, **the definition of the indicator function ($\mathbb{1}_X\equiv1$)**, Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $X=\biguplus_{j\in[0..m]}B_j$), Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I\stackrel{\text{def.}}{=}[0..m]$), **left distributivity of multiplication over addition in $\mathbb{R}$**, and Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*), we have

$$\mathbb{1}_{A_i}=\mathbb{1}_{A_i}\,\mathbb{1}_{\biguplus_{j\in[0..m]}B_j}=\mathbb{1}_{A_i}\left(\sum_{j\in[0..m]}\mathbb{1}_{B_j}\right)=\sum_{j\in[0..m]}(\mathbb{1}_{A_i}\,\mathbb{1}_{B_j})=\sum_{j\in[0..m]}\mathbb{1}_{A_i\cap B_j}.$$

Thus, from **left distributivity of multiplication over addition in $\mathbb{R}$**, **the definition of intersection ($A_i\cap B_j\subset A_i$)**, (2a), and Lemma 754 (*$\mathcal{SF}$ disjoint representation*), we have

$$f=\sum_{i\in[0..n]}a_i\left(\sum_{j\in[0..m]}\mathbb{1}_{A_i\cap B_j}\right)=\sum_{(i,j)\in[0..n]\times[0..m]}a_i\,\mathbb{1}_{A_i\cap B_j},$$

and the representation is disjoint. Similarly, we obtain the disjoint representation

$$g=\sum_{(i,j)\in[0..n]\times[0..m]}b_j\,\mathbb{1}_{A_i\cap B_j}.$$

Hence, from **additive group properties of $\mathbb{R}$**, **right distributivity of multiplication over addition in $\mathbb{R}$**, and Lemma 475 (*equivalent definition of σ-algebra*, closedness under intersection), we have Equation (13.21) where again, for all $k \in [0..N]$, $C_{\varphi(k)} \in \Sigma$.

(3a). Let $a, b \in \mathbb{R}$. Let $\diamondsuit$ be a binary operator from $\mathbb{R}^2$ to $\mathbb{R}$ (using infix notation). Then, from **the definition of inverse image**, we have $(f \diamondsuit g)(f^{-1}(a) \cap g^{-1}(b)) \subset \{a \diamondsuit b\}$. Hence, from **identity $A \subset \psi^{-1}(B) \Leftrightarrow \psi(A) \subset B$ (with $\psi \stackrel{\text{def.}}{=} f \diamondsuit g$)**, we have $f^{-1}(a) \cap g^{-1}(b) \subset (f \diamondsuit g)^{-1}(a \diamondsuit b)$.

(3b). Let $k \in [0..N]$. Let $(i, j) \stackrel{\text{def.}}{=} \varphi(k) \in [0..n] \times [0..m]$. Then, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $A_i \subset f^{-1}(a_i)$ and $B_j \subset g^{-1}(b_j)$), **monotonicity of intersection ($A_i \cap B_j$ is a subset of $f^{-1}(a_i) \cap g^{-1}(b_j)$)**, and (3a) with the binary operators addition and multiplication, we have $C_{\varphi(k)} \subset (f+g)^{-1}(c^+_{\varphi(k)})$ and $C_{\varphi(k)} \subset (fg)^{-1}(c^*_{\varphi(k)})$. Therefore, from Lemma 754 (*$\mathcal{SF}$ disjoint representation*), and (2a), both Equations (13.21) and (13.22) are disjoint representations. $\square$

Remark 758. Note that in the previous lemma, formula (13.21) is wrong when there is nonempty overlapping in the representations of the functions. Indeed, the values in overlapped parts are counted too many times.

Note also that even though the representations of f and g are canonical, the representations of $f + g$ and fg in Equations (13.21) and (13.22) are only disjoint. Indeed, the $a_i + b_j$'s and $a_i b_j$'s may not be pairwise distinct, and thus the $A_i \cap B_j$'s may no longer be their inverse images.

Lemma 759 ***($\mathcal{SF}$ is measurable).***
Let (X, Σ) be a measurable space. Then, we have $\mathcal{IF} \subset \mathcal{SF} \subset \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$.

Proof. Direct consequence of Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), Lemma 569 (*measurability of indicator function*), Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*), and Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$). $\square$

Lemma 760 ***($\mathcal{SF}$ is closed under extension by zero).***
Let (X, Σ) be a measurable space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$ and $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$ and $f_{|A} \in \mathcal{SF}(A, \Sigma \overline{\cap} A)$. Then, we have $\hat{f}\mathbb{1}_A \in \mathcal{SF}(X, \Sigma)$.

Proof. Direct consequence of Definition 748 (*$\mathcal{SF}$, vector space of simple functions*, $f_{|A}$ is a linear combination of f_i's in $\mathcal{IF}(A, \Sigma \overline{\cap} A)$), and Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*, with $\hat{f}_i$ function from X to $\overline{\mathbb{R}}$ such that $(\hat{f}_i)_{|Y} = f_i$). $\square$

Lemma 761 ***($\mathcal{SF}$ is closed under restriction).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{SF}(X, \Sigma)$. Let $A \in \Sigma$. Then, we have $f_{|A} \in \mathcal{SF}(A, \Sigma \overline{\cap} A)$.*

Proof. Direct consequence of Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), and Lemma 739 (*$\mathcal{IF}$ is closed under restriction*). $\square$

13.2.2 Nonnegative simple function

Remark 762. From now on, the functions take their values in $\mathbb{R}_+$, and the expressions involving integrals are taken in $\overline{\mathbb{R}}_+$.

Definition 763 ***($\mathcal{SF}_+$, subset of nonnegative simple functions).***
Let (X, Σ) be a measurable space. The *subset of nonnegative simple functions* is denoted $\mathcal{SF}_+(X, \Sigma)$ (or simply $\mathcal{SF}_+$); it is defined by $\mathcal{SF}_+ \stackrel{\text{def.}}{=} \{f \in \mathcal{SF} \,|\, f(X) \subset \mathbb{R}_+\}$.

Lemma 764 ***($\mathcal{SF}_+$ disjoint representation).***
Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$.
Then, $f \in \mathcal{SF}_+$ iff there exists $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}_+$, and $(A_i)_{i\in[0..n]} \in \Sigma$ such that

$$\forall i \in [0..n], \quad A_i \subset f^{-1}(a_i), \tag{13.23}$$
$$\forall p, q \in [0..n], \quad p \neq q \quad \Longrightarrow \quad A_p \cap A_q = \varnothing, \tag{13.24}$$
$$X = \biguplus_{i\in[0..n]} A_i, \tag{13.25}$$
$$f = \sum_{i\in[0..n]} a_i \, \mathbb{1}_{A_i} \tag{13.26}$$

Proof. **"Left" implies "right".** Direct consequence of Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), and Lemma 754 (*$\mathcal{SF}$ disjoint representation*, with $f(A_i) = \{a_i\} \subset \mathbb{R}_+$).

"Right" implies "left".
Direct consequence of Lemma 754 (*$\mathcal{SF}$ disjoint representation*, $f \in \mathcal{SF}$), **closedness of addition in $\mathbb{R}_+$ ($f \geqslant 0$)**, and Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*).

Therefore, we have the equivalence. □

Lemma 765 ***($\mathcal{SF}_+$ canonical representation).***
Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$.
Then, $f \in \mathcal{SF}_+$ iff there exists unique $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}_+$, and $(A_i)_{i\in[0..n]} \in \Sigma$ such that

$$\forall i \in [0..n-1], \quad a_i < a_{i+1}, \tag{13.27}$$
$$\forall i \in [0..n], \quad A_i = f^{-1}(a_i) \neq \varnothing, \tag{13.28}$$
$$\forall p, q \in [0..n], \quad p \neq q \quad \Longrightarrow \quad A_p \cap A_q = \varnothing, \tag{13.29}$$
$$X = \biguplus_{i\in[0..n]} A_i, \tag{13.30}$$
$$f = \sum_{i\in[0..n]} a_i \, \mathbb{1}_{A_i} \tag{13.31}$$

which can also be written (with $f^{-1}(y) \in \Sigma$ for all $y \in f(X)$)

$$f = \sum_{y\in f(X)} y \, \mathbb{1}_{f^{-1}(y)}. \tag{13.32}$$

Proof. **"Left" implies "right".** Direct consequence of Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), and Lemma 752 (*$\mathcal{SF}$ canonical representation*, with $a_i \in f(X) \subset \mathbb{R}_+$).

"Right" implies "left".
Direct consequence of Lemma 752 (*$\mathcal{SF}$ canonical representation*, $f \in \mathcal{SF}$), **closedness of addition in $\mathbb{R}_+$ ($f \geqslant 0$)**, and Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*).

Therefore, we have the equivalence. □

Lemma 766 ***($\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{SF}_+$. Let $n, m \in \mathbb{N}$, $(a_i)_{i\in[0..m]}, (b_j)_{j\in[0..m]} \in \mathbb{R}_+$, and $(A_i)_{i\in[0..m]}, (B_j)_{j\in[0..m]} \in \Sigma$. Assume that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$ is a disjoint representation, and that $f = \sum_{j\in[0..m]} b_j \mathbb{1}_{B_j}$ is the canonical representation.*

For all $j \in [0..m]$, let $I'_j \overset{\text{def.}}{=} \{i \in [0..n] \mid \varnothing \neq A_i \subset B_j\}$. Let $I' \overset{\text{def.}}{=} \{i \in [0..n] \mid A_i \neq \varnothing\}$. Then, we have

$$(13.33) \qquad \forall j \in [0..m], \quad B_j = \biguplus_{i\in I'_j} A_i \quad \wedge \quad (\forall i \in I'_j,\ a_i = b_j),$$

$$(13.34) \qquad (\forall p, q \in [0..m],\ p \neq q \Rightarrow I'_p \cap I'_q = \varnothing) \quad \wedge \quad \biguplus_{j\in[0..m]} I'_j = I'.$$

Proof. Direct consequence of Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*), Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*, $\underline{a_i \geqslant 0}$), and Lemma 765 (*$\mathcal{SF}_+$ canonical representation*, $\underline{b_i \geqslant 0}$). □

Lemma 767 ***($\mathcal{SF}_+$ simple representation).*** *Let (X, Σ) be a measurable space. Let $f : X \to \mathbb{R}$. Then, we have $f \in \mathcal{SF}_+$ iff there exists $n \in \mathbb{N}$, $(a_i)_{i\in[0..n]} \in \mathbb{R}_+$, and $(A_i)_{i\in[0..n]}$ in Σ such that, $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$.*

If so, for all $x \in X \setminus \bigcup_{i\in[0..n]} A_i$, we have $f(x) = 0$.

Proof. **"Left" implies "right".**
Direct consequence of Lemma 765 (*$\mathcal{SF}_+$ canonical representation*).

"Right" implies "left".
Direct consequence of Lemma 749 (*$\mathcal{SF}$ simple representation*, $\underline{f \in \mathcal{SF}}$), **closedness of addition in $\mathbb{R}_+$ ($f \geqslant 0$)**, and Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*).

Therefore, we have the equivalence.

Property. Direct consequence of Lemma 749 (*$\mathcal{SF}$ simple representation*). □

Lemma 768 ***($\mathcal{SF}_+$ is closed under positive algebra operations).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Let $a \in \mathbb{R}_+$. Then, we have $f + g, af, fg \in \mathcal{SF}_+$.*

Let $f, g \in \mathcal{SF}_+$, $n, m \in \mathbb{N}$, $(a_i)_{i\in[0..n]}, (b_j)_{j\in[0..m]} \in \mathbb{R}_+$, and $(A_i)_{i\in[0..n]}, (B_j)_{j\in[0..m]} \in \Sigma$ such that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$ and $g = \sum_{j\in[0..m]} b_j \mathbb{1}_{B_j}$.

Let $N \overset{\text{def.}}{=} nm + n + m$. Let φ be a bijection from $[0..N]$ to $[0..n] \times [0..m]$.

For all $(i, j) \in [0..n] \times [0..m]$, let $c^+_{i,j} \overset{\text{def.}}{=} a_i + b_j$, $c^*_{i,j} \overset{\text{def.}}{=} a_i b_j$, and $C_{i,j} \overset{\text{def.}}{=} A_i \cap B_j$. Then, on the one hand, if both representations are disjoint, we have

$$(13.35) \qquad f + g = \sum_{k\in[0..N]} c^+_{\varphi(k)} \mathbb{1}_{C_{\varphi(k)}},$$

and it is also a disjoint representation. On the other hand, we always have

$$(13.36) \qquad fg = \sum_{k\in[0..N]} c^*_{\varphi(k)} \mathbb{1}_{C_{\varphi(k)}},$$

and it is also a disjoint representation when those of f and g are.

Proof. **Closedness.** Direct consequence of Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*), Definition 226 (*algebra over a field*), and **closedness of addition and multiplication in $\mathbb{R}_+$**.

Identities. Direct consequences of Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*), **closedness of addition and multiplication in $\mathbb{R}_+$**, Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*), and **associativity and commutativity of intersection**. □

Lemma 769 ***($\mathcal{SF}_+$ is measurable).***
Let (X, Σ) be a measurable space. Then, we have $\mathcal{IF} \subset \mathcal{SF}_+ \subset \mathcal{M}_\mathbb{R} \cap \mathcal{M}_+ \subset \mathcal{M}$.

Proof. Direct consequence of Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*), **nonnegativeness of the indicator function**, Lemma 759 (*$\mathcal{SF}$ is measurable*), Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*). □

13.2.3 Integration of nonnegative simple function

Lemma 770 ***(integral in $\mathcal{SF}_+$).*** *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{SF}_+$. Then, we have $f = \sum_{y \in f(X)} y \, \mathbb{1}_{f^{-1}(y)}$, and the sum $\sum_{y \in f(X)} y \, \mu(f^{-1}(y))$ is well-defined in $\overline{\mathbb{R}}_+$.*
The integral of f (for the measure μ) *is still denoted $\int f \, d\mu$; it is defined by*

$$\int f \, d\mu \stackrel{\text{def.}}{=} \sum_{y \in f(X)} y \, \mu\left(f^{-1}(y)\right) \quad \in \overline{\mathbb{R}}_+. \tag{13.37}$$

Proof. Direct consequence of Lemma 765 (*$\mathcal{SF}_+$ canonical representation*), Definition 611 (*measure*, nonnegativeness), Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*, $a_i \in \mathbb{R}_+ \subset \overline{\mathbb{R}}_+$), and Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*). □

Lemma 771 ***(integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$).***
*Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{IF}$. Then, the values of $\int f \, d\mu$ provided by Definition 740 (*integral in $\mathcal{IF}$*), and Lemma 770 (*integral in $\mathcal{SF}_+$*) coincide.*
In other terms, for all $A \in \Sigma$, we have $\mathbb{1}_A \in \mathcal{SF}_+$ and $\int \mathbb{1}_A \, d\mu = \mu(A)$.

Proof. Direct consequence of Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*, $A \in \Sigma$), Lemma 767 (*$\mathcal{SF}_+$ simple representation*, with $n = 0$, $a_0 = 1$, and $A_0 = A$), Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*), and Lemma 770 (*integral in $\mathcal{SF}_+$*, with $f(X) = \{0, 1\}$, $f^{-1}(0) = A^c$, and $f^{-1}(1) = A$). □

Lemma 772 ***(equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{SF}_+$. Let $n \in \mathbb{N}$, $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$, and $(A_i)_{i \in [0..n]} \in \Sigma$. Assume that $f = \sum_{i \in [0..n]} a_i \, \mathbb{1}_{A_i}$ is a disjoint representation. Then, we have

$$\int f \, d\mu = \sum_{i \in [0..n]} a_i \, \mu(A_i). \tag{13.38}$$

Proof. From Lemma 765 (*$\mathcal{SF}_+$ canonical representation*), let $m \in \mathbb{N}$, $(b_j)_{j \in [0..m]} \in \mathbb{R}_+$, and $(B_j)_{j \in [0..m]} \in \Sigma$ such that $f = \sum_{j \in [0..m]} b_j \, \mathbb{1}_{B_j}$ is the canonical representation.
For all $j \in [0..m]$, let $I'_j \stackrel{\text{def.}}{=} \{i \in [0..n] \,|\, \varnothing \neq A_i \subset B_j\}$. Let $I' \stackrel{\text{def.}}{=} \{i \in [0..n] \,|\, A_i \neq \varnothing\}$.
Then, from Lemma 770 (*integral in $\mathcal{SF}_+$*), Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*, both properties), Definition 611 (*measure*, σ-additivity and $\mu(\varnothing) = 0$), Definition 608 (*σ-additivity over measurable space*, with $I \stackrel{\text{def.}}{=} I'_j$), and **left distributivity of multiplication over addition in** $\mathbb{R}$, we have

$$\begin{aligned} \int f \, d\mu &= \sum_{j \in [0..m]} b_j \, \mu(B_j) = \sum_{j \in [0..m]} b_j \, \mu\left(\biguplus_{i \in I'_j} A_i\right) \\ &= \sum_{j \in [0..m]} \sum_{i \in I'_j} a_i \, \mu(A_i) = \sum_{i \in I'} a_i \, \mu(A_i) = \sum_{i \in [0..n]} a_i \, \mu(A_i). \end{aligned}$$

□

Remark 773. The next Lemmas 774 and 778 state the same result (additivity of the integral in $\mathcal{SF}_+$), but their proofs are different: the first one relies on the disjoint representation of simple functions, whereas the second one sticks to the canonical representation and uses a somewhat tedious change of variables.

Lemma 774 ***(integral in $\mathcal{SF}_+$ is additive).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Then, $f + g \in \mathcal{SF}_+$, and we have

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu. \tag{13.39}$$

Proof. From Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*), we have $f+g \in \mathcal{SF}_+$.

From Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*), let $n, m \in \mathbb{N}$, $(a_i)_{i\in[0..n]}, (b_j)_{j\in[0..m]} \in \mathbb{R}$ and $(A_i)_{i\in[0..n]}, (B_j)_{j\in[0..m]} \in \Sigma$ such that $f = \sum_{i\in[0..n]} a_i \mathbb{1}_{A_i}$ and $g = \sum_{j\in[0..m]} b_j \mathbb{1}_{B_j}$ are disjoint representations. In particular, we have $X = \biguplus_{i\in[0..n]} A_i = \biguplus_{j\in[0..m]} B_j$.

Let $N \stackrel{\text{def.}}{=} nm + n + m$. Since $N + 1 = (n+1)(m+1)$, there exists a bijection φ from $[0..N]$ to $[0..n] \times [0..m]$. For all $(i, j) \in [0..n] \times [0..m]$, let $c_{i,j} \stackrel{\text{def.}}{=} a_i + b_j$ and $C_{i,j} \stackrel{\text{def.}}{=} A_i \cap B_j$. Then, from Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*, addition), the representation $f + g = \sum_{k\in[0..N]} c_{\varphi(k)} \mathbb{1}_{C_{\varphi(k)}}$ is also disjoint. Hence, from Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*), **associativity and commutativity of addition in $\mathbb{R}$ ($\sum_k = \sum_{i,j}$)**, Lemma 613 (*measure over countable pseudopartition*, first with A_i and the pseudopartition $(B_j)_{j\in[0..m]}$, then with B_j and the pseudopartition $(A_i)_{i\in[0..n]}$), and **left distributivity of multiplication over addition in $\mathbb{R}$**, we have

$$\int (f+g)\, d\mu = \sum_{k\in[0..N]} c_{\varphi(k)}\, \mu(C_{\varphi(k)}) = \sum_{(i,j)\in[0..n]\times[0..m]} c_{i,j}\, \mu(C_{i,j}),$$
$$\int f\, d\mu = \sum_{i\in[0..n]} a_i\, \mu(A_i) = \sum_{(i,j)\in[0..n]\times[0..m]} a_i\, \mu(C_{i,j}),$$
$$\int g\, d\mu = \sum_{j\in[0..m]} b_j\, \mu(B_j) = \sum_{(i,j)\in[0..n]\times[0..m]} b_j\, \mu(C_{i,j}).$$

Therefore, from **left distributivity of multiplication over addition in $\mathbb{R}$**, we have the equality. □

Lemma 775 ***(decomposition of measure in $\mathcal{SF}_+$).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Let $y \in f(X)$. Then, we have

$$\mu\left(f^{-1}(y)\right) = \sum_{z\in g(X)} \mu\left(f^{-1}(y) \cap g^{-1}(z)\right). \tag{13.40}$$

Proof. Direct consequence of Lemma 769 (*$\mathcal{SF}_+$ is measurable*), Lemma 518 (*some Borel subsets*, singletons are measurable), Lemma 571 (*inverse image is measurable in $\mathbb{R}$*, $f^{-1}(y), g^{-1}(z) \in \Sigma$), Lemma 752 (*$\mathcal{SF}$ canonical representation*, $(g^{-1}(z))_{z\in g(X)}$ form a partition of X), and Lemma 613 (*measure over countable pseudopartition*, with $A \stackrel{\text{def.}}{=} f^{-1}(y)$, $B_i \stackrel{\text{def.}}{=} g^{-1}(z)$ and $\mathrm{card}(I)$ equals $\mathrm{card}(g(X))$). □

Lemma 776 ***(change of variable in sum in $\mathcal{SF}_+$).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Let $y \in f(X)$. Then, we have

$$\sum_{z\in g(X)} (y+z)\, \mu\left(f^{-1}(y) \cap g^{-1}(z)\right) = \sum_{t\in(f+g)(X)} t\, \mu\left(f^{-1}(y) \cap (f+g)^{-1}(t)\right). \tag{13.41}$$

Proof. For all $z, t \in \mathbb{R}$, let $A(z) \stackrel{\text{def.}}{=} f^{-1}(y) \cap g^{-1}(z)$ and $B(t) \stackrel{\text{def.}}{=} f^{-1}(y) \cap (f+g)^{-1}(t)$. Then, from Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*), and Lemma 769 (*$\mathcal{SF}_+$ is measurable*), we have $f, g, f+g \in \mathcal{SF}_+ \subset \mathcal{M}_+ \subset \mathcal{M}$.

Let $z, t \in \mathbb{R}$. Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Lemma 571 (*inverse image is measurable in* $\mathbb{R}$, $f^{-1}(y), g^{-1}(z), (f+g)^{-1}(t) \in \Sigma$), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have $A(z), B(t) \in \Sigma$.

Let $\mathcal{A} \stackrel{\text{def.}}{=} \{z \in g(X) \,|\, \mu(A(z)) > 0\}$ and $\mathcal{B} \stackrel{\text{def.}}{=} \{t \in (f+g)(X) \,|\, \mu(B(t)) > 0\}$.

(1). $\forall z \in g(X)$, $A(z) = B(y+z)$.

Direct consequence of the definition of $A(z)$ and $B(y+z)$, **the definition of the addition of functions to** $\mathbb{R}$, and **additive abelian group properties of** $\mathbb{R}$ **(with $y \neq \infty$)**.

(2). $\forall t \in (f+g)(X)$, $B(t) = A(-y+t)$.

Direct consequence of the definition of $B(t)$ and $A(-y+t)$, **the definition of the addition of functions to** $\mathbb{R}$, and **additive abelian group properties of** $\mathbb{R}$ **(with $y \neq \infty$)**.

(3). $\tau_y = (z \mapsto y+z) : \mathcal{A} \to \mathcal{B}$ is a bijection.

From **additive abelian group properties of** $\mathbb{R}$ **(with $y \neq \infty$)**, the translation τ_y is a bijection from $\mathbb{R}$ onto itself.

Let $z \in \mathcal{A}$. Then, from the definition of $\mathcal{A}$, we have $z \in g(X)$ and $\mu(A(z)) > 0$. Thus, from Definition 611 (*measure*, $\mu(\varnothing) = 0$ (contrapositive)), we have $A(z) \neq \varnothing$. Let $x \in A(z)$. Then, from the definition of $A(z)$, we have $f(x) = y$ and $g(x) = z$. Thus, from the definition of the addition of functions to $\mathbb{R}$, we have $(f+g)(x) = y+z$, i.e. $y+z \in (f+g)(X)$. Moreover, from (1), we have $\mu(B(y+z)) = \mu(A(z)) > 0$. Thus, from the definition of $\mathcal{B}$, we have $\tau_y(z) = y+z \in \mathcal{B}$. Hence, we have $\tau_y(\mathcal{A}) \subset \mathcal{B}$.

Conversely, let $t \in \mathcal{B}$. Then, from the definition of $\mathcal{B}$, we have $t \in (f+g)(X)$ and $\mu(B(t)) > 0$. Thus, from Definition 611 (*measure*, $\mu(\varnothing) = 0$ (contrapositive)), we have $B(t) \neq \varnothing$. Let $x \in B(t)$. Then, from the definition of $B(t)$, we have $f(x) = y$ and $(f+g)(x) = t$. Thus, from the definition of the addition of functions to $\mathbb{R}$, and **additive abelian group properties of** $\mathbb{R}$ **(with $y \neq \infty$)**, we have $g(x) = -y+t$, i.e. $-y+t \in g(X)$. Moreover, from (2), we have $\mu(A(-y+t)) = \mu(B(t)) > 0$. Thus, from the definition of $\mathcal{A}$, we have $\tau_y^{-1}(t) = -y+t \in \mathcal{A}$. Then, from **properties of inverse functions**, we have $t = \tau_y(\tau_y^{-1}(t)) \in \tau_y(\mathcal{A})$. Hence, we have $\mathcal{B} \subset \tau_y(\mathcal{A})$.

Finally, τ_y is a bijection from $\mathcal{A}$ onto $\tau_y(\mathcal{A}) = \mathcal{B}$.

Therefore, from the definitions of $\mathcal{A}$ and τ_y, (1), (3), and the definition of $\mathcal{B}$, we have

$$\begin{aligned}\sum_{z \in g(X)} (y+z)\,\mu(A(z)) &= \sum_{z \in \mathcal{A}} (y+z)\,\mu(A(z)) = \sum_{z \in \mathcal{A}} \tau_y(z)\,\mu(B(\tau_y(z))) \\ &= \sum_{t \in \tau_y(\mathcal{A})} t\,\mu(B(t)) = \sum_{t \in \mathcal{B}} t\,\mu(B(t)) = \sum_{t \in (f+g)(X)} t\,\mu(B(t)).\end{aligned}$$

□

Remark 777. Note that the previous lemma is still valid when f and g are functions with possibly changing sign, and $y \in \mathbb{R}$. Moreover, the equalities $A(z) = B(y+z)$ and $B(t) = A(-y+t)$ in the proof are still valid for any $z, t \in \mathbb{R}$, in which case these subsets may be empty.

Lemma 778 ***(integral in $\mathcal{SF}_+$ is additive (alternate proof)).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Then, $f+g \in \mathcal{SF}_+$, and we have

$$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu. \tag{13.42}$$

Proof. From Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*), and Lemma 769 (*$\mathcal{SF}_+$ is measurable*), we have $f, g, f+g \in \mathcal{SF}_+ \subset \mathcal{M}_+ \subset \mathcal{M}$.

Let $y, z, t \in \mathbb{R}$. Then, from Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*), Lemma 571 (*inverse image is measurable in* $\mathbb{R}$, $\underline{f^{-1}(y), g^{-1}(z), (f+g)^{-1}(t) \in \Sigma}$), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection), we have

$$f^{-1}(y) \cap g^{-1}(z), f^{-1}(y) \cap (f+g)^{-1}(t) \in \Sigma.$$

From Lemma 770 (*integral in $\mathcal{SF}_+$*), Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*), and Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*), we have

$$\int f \, d\mu = \sum_{y \in f(X)} y \, \mu\left(f^{-1}(y)\right) = \sum_{y \in f(X)} \sum_{z \in g(X)} y \, \mu\left(f^{-1}(y) \cap g^{-1}(z)\right).$$

In the very same way, we have

$$\int g \, d\mu = \sum_{z \in g(X)} z \, \mu\left(g^{-1}(z)\right) = \sum_{z \in g(X)} \sum_{y \in f(X)} z \, \mu\left(g^{-1}(z) \cap f^{-1}(y)\right).$$

Therefore, from Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*), Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*), **commutativity of intersection**, Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*), Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*), Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*, with $(f+g)$, f and t), and Lemma 770 (*integral in $\mathcal{SF}_+$*), we have

$$\begin{aligned}
\int f \, d\mu + \int g \, d\mu &= \sum_{y \in f(X)} \sum_{z \in g(X)} (y+z) \, \mu\left(f^{-1}(y) \cap g^{-1}(z)\right) \\
&= \sum_{y \in f(X)} \sum_{t \in (f+g)(X)} t \, \mu\left(f^{-1}(y) \cap (f+g)^{-1}(t)\right) \\
&= \sum_{t \in (f+g)(X)} t \sum_{y \in f(X)} \mu\left((f+g)^{-1}(t) \cap f^{-1}(y)\right) \\
&= \sum_{t \in (f+g)(X)} t \, \mu\left((f+g)^{-1}(t)\right) \\
&= \int (f+g) \, d\mu.
\end{aligned}$$

□

Lemma 779 *(integral in $\mathcal{SF}_+$ is positive linear).* *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{SF}_+$. Let $a \in \mathbb{R}_+$. Then, $f+g, af \in \mathcal{SF}_+$, and we have*

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu \quad \text{and} \quad \int af \, d\mu = a \int f \, d\mu. \tag{13.43}$$

Proof. **Addition.** Direct consequence of Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*), or Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Nonnegative scalar multiplication.
From Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*), we have $af \in \mathcal{SF}_+$.

Case $a = 0$. Then, from **field properties of $\mathbb{R}$ ($0f \equiv 0$)**, and **the definition of the indicator function ($\mathbb{1}_\varnothing \equiv 0$)**, we have $0f = \mathbb{1}_\varnothing$. Hence, from Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*), Definition 611 (*measure*, homogeneity), and Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), we have

$$\int 0f \, d\mu = \int \mathbb{1}_\varnothing \, d\mu = \mu(\varnothing) = 0 = 0 \times \int f \, d\mu.$$

Case $a > 0$. Let $y \in f(X)$. Let $z = ay \in (af)(X)$. Then, from Definition 288 (*multiplication in* $\overline{\mathbb{R}}$, for all $a \in \mathbb{R}_+^\star$, $\frac{a}{a} = 1$), we have $(af)^{-1}(z) = f^{-1}(y)$ Hence, from Lemma 770 (*integral in* $\mathcal{SF}_+$), we have

$$\int af\,d\mu = \sum_{z \in (af)(X)} z\,\mu((af)^{-1}(z)) = \sum_{y \in f(X)} ay\,\mu(f^{-1}(y)) = a \int f\,d\mu.$$

□

Lemma 780 ***(equivalent definition of the integral in*** $\mathcal{SF}_+$ ***(simple)).***
Let (X, Σ, μ) *be a measure space. Let* $f \in \mathcal{SF}_+$*. Let* $n \in \mathbb{N}$*,* $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$*, and* $(A_i)_{i \in [0..n]} \in \Sigma$*. Assume that* $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ *is a simple representation. Then, we have*

$$\int f\,d\mu = \sum_{i \in [0..n]} a_i\,\mu(A_i). \tag{13.44}$$

Proof. Direct consequence of Lemma 767 ($\mathcal{SF}_+$ *simple representation*, thus such representation exists), Lemma 779 (*integral in* $\mathcal{SF}_+$ *is positive linear*), and Lemma 771 (*integral in* $\mathcal{SF}_+$ *generalizes integral in* $\mathcal{IF}$). □

Lemma 781 ***(integral in*** $\mathcal{SF}_+$ ***is monotone).***
Let (X, Σ, μ) *be a measure space. Let* $f, g \in \mathcal{SF}_+$*. Then, we have*

$$f \leqslant g \quad \Longrightarrow \quad \int f\,d\mu \leqslant \int g\,d\mu. \tag{13.45}$$

Proof. Assume that $f \leqslant g$. Then, from Definition 763 ($\mathcal{SF}_+$*, subset of nonnegative simple functions*), Lemma 757 ($\mathcal{SF}$ *is algebra over* $\mathbb{R}$), Definition 226 (*algebra over a field*, $\mathcal{SF}$ is a vector space), and Definition 61 (*vector space*, $(\mathcal{SF}, +)$ is an abelian group), we have $g = f + (g - f)$ with $f, g - f \in \mathcal{SF}_+$. Therefore, from Lemma 779 (*integral in* $\mathcal{SF}_+$ *is positive linear*), and Lemma 770 (*integral in* $\mathcal{SF}_+$, nonnegativeness with $g - f \in \mathcal{SF}_+$), we have

$$\int g\,d\mu = \int f\,d\mu + \int (g - f)\,d\mu \geqslant \int f\,d\mu.$$

□

Lemma 782 ***(integral in*** $\mathcal{SF}_+$ ***is continuous).***
Let (X, Σ, μ) *be a measure space. Let* $f \in \mathcal{SF}_+$*. Then, we have*

$$\int f\,d\mu = \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leqslant f}} \int \varphi\,d\mu. \tag{13.46}$$

Proof. From Definition 2 (*supremum*, upper bound), and since $f \in \mathcal{SF}_+$, we have

$$\int f\,d\mu \leqslant \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leqslant f}} \int \varphi\,d\mu.$$

Conversely, let $\varphi \in \mathcal{SF}_+$ such that $\varphi \leqslant f$. Then, from Lemma 781 (*integral in* $\mathcal{SF}_+$ *is monotone*), and Definition 2 (*supremum*, least upper bound), we have

$$\sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leqslant f}} \int \varphi\,d\mu \leqslant \int f\,d\mu.$$

Therefore, we have the equality. □

Lemma 783 *(integral in $\mathcal{SF}_+$ over subset).* *Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, we have $f_{|A} \in \mathcal{SF}_+(A, \Sigma \sqcap A)$ iff $\hat{f}\,\mathbb{1}_A \in \mathcal{SF}_+(X, \Sigma)$. If so, we have*

$$\int f_{|A}\, d\mu_A = \int \hat{f}\,\mathbb{1}_A\, d\mu. \tag{13.47}$$

This integral is still denoted $\int_A f\, d\mu$; it is still called integral of f over A.

Proof. **Equivalence.** Direct consequence of Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*), and Lemma 761 (*$\mathcal{SF}$ is closed under restriction*).

Identity. Direct consequence of Definition 748 (*$\mathcal{SF}$, vector space of simple functions*, $f_{|A}$ is a linear combination of f_i's in $\mathcal{IF}(A, \Sigma \sqcap A)$) Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*), Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*), and Lemma 743 (*integral in $\mathcal{IF}$ over subset*, with $\hat{f}_i : X \to \overline{\mathbb{R}}$ such that $(\hat{f}_i)_{|Y} = f_i$). □

Lemma 784 *(integral in $\mathcal{SF}_+$ over subset is additive).* *Let (X, Σ, μ) be a measure space. Let $n \in \mathbb{N}$. Let $A, (A_i)_{i\in[0..n]} \in \Sigma$. Assume that $(A_i)_{i\in[0..n]}$ is a pseudopartition of A. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$.*
Then, $\hat{f}\,\mathbb{1}_A \in \mathcal{SF}_+$ iff for all $i \in [0..n]$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{SF}_+$. If so, we have

$$\int_A f\, d\mu = \sum_{i\in[0..n]} \int_{A_i} f\, d\mu. \tag{13.48}$$

Proof. **(1).** Let $W \in \Sigma$. Assume that $W \subset Y$. Then, from Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), and Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), we have $\hat{f}\,\mathbb{1}_W \in \mathcal{SF}_+$ iff there exist $p \in \mathbb{N}$, $(a_j)_{j\in[0..p]} \in \mathbb{R}_+$, and $(f_j)_{j\in[0..p]} \in \mathcal{IF}$ such that

$$\hat{f}\,\mathbb{1}_W = \sum_{j\in[0..p]} a_j f_j.$$

(2). Let $g \in \mathcal{IF}$. Let $i \in [0..n]$. Then, from Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*, $g = \mathbb{1}_B$ with $B \in \Sigma$), and Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*), we have $g\,\mathbb{1}_{A_i} = \mathbb{1}_{B\cap A_i} \in \mathcal{IF}$.

"Left" implies "right". Assume that $\hat{f}\,\mathbb{1}_A \in \mathcal{SF}_+$.
Then, from (1) (with $W \stackrel{\text{def.}}{=} A$), let $p \in \mathbb{N}$, $(a_j)_{j\in[0..p]} \in \mathbb{R}_+$, and $(f_j)_{j\in[0..p]} \in \mathcal{IF}$ such that

$$\hat{f}\,\mathbb{1}_A = \sum_{j\in[0..p]} a_j f_j.$$

Let $i \in [0..n]$. Let $j \in [0..p]$. Let $g_j^i \stackrel{\text{def.}}{=} f_j\,\mathbb{1}_{A_i}$. Then, from (2) (with $g \stackrel{\text{def.}}{=} f_j$), we have $g_j^i \in \mathcal{IF}$. Moreover, since $A_i \subset A$ (i.e. $A_i = A \cap A_i$), from **field properties of** $\mathbb{R}$, and Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*) we have

$$\hat{f}\,\mathbb{1}_{A_i} = \hat{f}\,\mathbb{1}_{A\cap A_i} = \left(\hat{f}\,\mathbb{1}_A\right)\mathbb{1}_{A_i} = \sum_{j\in[0..p]} a_j f_j\,\mathbb{1}_{A_i} = \sum_{j\in[0..p]} a_j g_j^i.$$

Hence, from (1) (with $W \stackrel{\text{def.}}{=} A_i$), we have $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{SF}_+$.

"Right" implies "left". Assume that for all $i \in [0..n]$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{SF}_+$.
Then, from (1) (with $W \stackrel{\text{def.}}{=} A_i$), for all $i \in [0..n]$, let $p^i \in \mathbb{N}$, $(a_{j^i}^i)_{j^i\in[0..p^i]} \in \mathbb{R}_+$, and $(f_{j^i}^i)_{j^i\in[0..p^i]}$

in $\mathcal{IF}$ such that $\hat{f}\,\mathbb{1}_{A_i} = \sum_{j^i\in[0..p^i]} a^i_{j^i} f^i_j$. Let $p \stackrel{\text{def.}}{=} -1 + \sum_{i\in[0..n]}(1+p^i)$,

$$(a_j)_{j\in[0..p]} \stackrel{\text{def.}}{=} \left((a^i_{j^i})_{j^i\in[0..p^i]}\right)_{i\in[0..n]} \in \mathbb{R}_+ \quad\text{and}\quad (f_j)_{j\in[0..p]} \stackrel{\text{def.}}{=} \left((f^i_{j^i}\,\mathbb{1}_{A_i})_{j^i\in[0..p^i]}\right)_{i\in[0..n]} \in \mathcal{IF}.$$

Then, from Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I \stackrel{\text{def.}}{=} [0..n]$), and **field properties of $\mathbb{R}$**, and (2) (with $g \stackrel{\text{def.}}{=} f^i_{j^i}$), we have

$$\hat{f}\,\mathbb{1}_A = \sum_{i\in[0..n]} \hat{f}\,\mathbb{1}_{A_i} = \sum_{i\in[0..n]}\sum_{j^i\in[0..p^i]} a^i_{j^i}(f^i_j\,\mathbb{1}_{A_i}) = \sum_{j\in[0..p]} a_j f_j.$$

Hence, from (1) (with $W \stackrel{\text{def.}}{=} A$), we have $\hat{f}\,\mathbb{1}_A \in \mathcal{SF}_+$.

Therefore, we have the equivalence.

Identity. Direct consequence of Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*, with A and the A_i's), and Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I \stackrel{\text{def.}}{=} [0..n]$). □

Lemma 785 ***(integral in $\mathcal{SF}_+$ for counting measure).***
Let (X,Σ) be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{SF}_+$. Then, we have

$$\int f\,d\delta_Y = \sum_{y\in Y} f(y). \tag{13.49}$$

Proof. Direct consequence of Lemma 767 (*$\mathcal{SF}_+$ simple representation*), Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*), Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*), and **associativity and commutativity of (possibly uncountable) addition in $\mathbb{R}_+$**. □

Lemma 786 ***(integral in $\mathcal{SF}_+$ for counting measure on $\mathbb{N}$).***
Let $f \in \mathcal{SF}_+(\mathbb{N},\mathcal{P}(\mathbb{N}))$. Then, we have

$$\int f\,d\delta_{\mathbb{N}} = \sum_{n\in\mathbb{N}} f(n). \tag{13.50}$$

Proof. Direct consequence of Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*, with $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$ and $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$). □

Lemma 787 ***(integral in $\mathcal{SF}_+$ for Dirac measure).***
Let (X,Σ) be a measurable space. Let $\{a\} \in \Sigma$. Let $f \in \mathcal{SF}_+$. Then, we have

$$\int f\,d\delta_a = f(a). \tag{13.51}$$

Proof. Direct consequence of Definition 675 (*Dirac measure*), and Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*, with $Y \stackrel{\text{def.}}{=} \{a\}$). □

13.3 Integration of nonnegative measurable functions

Remark 788. In this section, functions take their values in $\overline{\mathbb{R}}_+$, and the expressions involving integrals are also taken in $\overline{\mathbb{R}}_+$.

Lemma 789 ***(integral in $\mathcal{M}_+$).*** *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}_+$. Then, $\{\int \varphi \, d\mu \,|\, \varphi \in \mathcal{SF}_+ \wedge \varphi \leqslant f\}$ admits a supremum.*

The integral of f (for the measure μ) *is still denoted $\int f \, d\mu$; it is defined by*

$$\int f \, d\mu \stackrel{\text{def.}}{=} \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leqslant f}} \int \varphi \, d\mu \quad \in \overline{\mathbb{R}}_+. \tag{13.52}$$

A function $f : X \to \overline{\mathbb{R}}_+$ is said μ-integrable (in $\mathcal{M}_+$) *iff $f \in \mathcal{M}_+$ and $\int f \, d\mu < \infty$.*

Proof. Direct consequence of Lemma 770 (*integral in $\mathcal{SF}_+$*), and Definition 2 (*supremum*, with the Lebesgue integral in $\mathcal{SF}_+$ over $\{\varphi \in \mathcal{SF}_+ \,|\, \varphi \leqslant f\}$), □

Lemma 790 ***(integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$).*** *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{SF}_+$. Then, the values of $\int f \, d\mu$ provided by Lemma 770 (*integral in $\mathcal{SF}_+$*), and Lemma 789 (*integral in $\mathcal{M}_+$*) coincide.*

Proof. Direct consequence of Lemma 770 (*integral in $\mathcal{SF}_+$*), Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*), and Lemma 789 (*integral in $\mathcal{M}_+$*). □

Lemma 791 ***(integral in $\mathcal{M}_+$ of indicator function).*** *Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Then, we have $\mathbb{1}_A \in \mathcal{M}_+$ and $\int \mathbb{1}_A \, d\mu = \mu(A)$.*

Proof. Direct consequence of Lemma 769 (*$\mathcal{SF}_+$ is measurable*), Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*), and Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*). □

Lemma 792 ***(integral in $\mathcal{M}_+$ is positive homogeneous).*** *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}_+$. Let $a \in \mathbb{R}_+$. Then, $af \in \mathcal{M}_+$, and we have*

$$\int af \, d\mu = a \int f \, d\mu. \tag{13.53}$$

Proof. From Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*), $af \in \mathcal{M}_+$.

Case $a = 0$. Then, from Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), and Definition 61 (*vector space*, $0 \in \mathcal{SF}$), we have

$$0f = 0 \in \mathcal{SF}_+ \subset \mathcal{M}_+.$$

Hence, from Lemma 789 (*integral in $\mathcal{M}_+$*), Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*), Lemma 770 (*integral in $\mathcal{SF}_+$*), and Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), we have

$$\int 0f \, d\mu = \int 0 \, d\mu = 0 = 0 \times \int f \, d\mu.$$

Case $a > 0$. Then, from Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*), and Definition 288 (*multiplication in $\overline{\mathbb{R}}$*, for all $a \in \mathbb{R}_+^\star$, $\frac{a}{a} = 1$), we have $(\star)$ $\forall b \in \overline{\mathbb{R}}_+$, $\frac{1}{a}(ab) = b$.

Let $\varphi \in \mathcal{SF}_+$ such that $\varphi \leqslant f$. Then, from Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*), we have $a\varphi \in \mathcal{SF}_+$. Moreover, from **compatibility of multiplication by a positive number with order in $\overline{\mathbb{R}}$**, property $(\star)$, Lemma 779 (*integral in $\mathcal{SF}_+$ is positive* *linear*, positive homogeneity), Lemma 789 (*integral in $\mathcal{M}_+$*), and Definition 2 (*supremum*, upper bound), we have $a\varphi \leqslant af$ and

$$\int \varphi \, d\mu = \frac{1}{a} \int a\varphi \, d\mu \leqslant \frac{1}{a} \int af \, d\mu.$$

Thus, from Lemma 789 (*integral in $\mathcal{M}_+$*), Definition 2 (*supremum*, least upper bound), and property ($\star$), we have

$$a \int f \, d\mu = a \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leqslant f}} \int \varphi \, d\mu \leqslant a \frac{1}{a} \int af \, d\mu = \int af \, d\mu.$$

Moreover, the same result also holds for function $af \in \mathcal{M}_+$ and number $\frac{1}{a} > 0$, and with property ($\star$), we have $\int af \, d\mu \leqslant a \int \frac{1}{a}(af) \, d\mu = a \int f \, d\mu$. Hence, we have the equality.

Therefore, we always have the equality. □

Lemma 793 ***(integral in $\mathcal{M}_+$ of zero is zero).***
Let (X, Σ, μ) be a measure space. Then, $0 \in \mathcal{M}_+$, and we have $\int 0 \, d\mu = 0$.

Proof. Direct consequence of Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*, with $a \overset{\text{def.}}{=} 0$), Definition 288 (*multiplication in $\overline{\mathbb{R}}$*), and Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*). □

Lemma 794 ***(integral in $\mathcal{M}_+$ is monotone).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}_+$. Then, we have

$$f \leqslant g \quad \Longrightarrow \quad \int f \, d\mu \leqslant \int g \, d\mu. \tag{13.54}$$

Proof. Assume that $f \leqslant g$. Then, $\{\varphi \in \mathcal{SF}_+ \,|\, \varphi \leqslant f\} \subset \{\varphi \in \mathcal{SF}_+ \,|\, \varphi \leqslant g\}$. Therefore, from Lemma 789 (*integral in $\mathcal{M}_+$*), and **monotonicity of supremum**, we have

$$\int f \, d\mu \leqslant \int g \, d\mu.$$

□

Remark 795. The next proof follows step 3 of the Lebesgue scheme (see Section 4.1).
See also the sketch of the proof in Section 5.5.

Theorem 796 ***(Beppo Levi, monotone convergence).***
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$. Assume that the sequence is pointwise nondecreasing. Then, $\lim_{n \to \infty} f_n \in \mathcal{M}_+$, and we have

$$\int \lim_{n \to \infty} f_n \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu. \tag{13.55}$$

Proof. From **convergence of monotone sequences in $\overline{\mathbb{R}}$**, and **completeness of $\overline{\mathbb{R}}$**, the existence of the pointwise limit f is guaranteed in $\overline{\mathbb{R}}$, and we have $f = \lim_{n \to \infty} f_n = \sup_{n \in \mathbb{N}} f_n$.

From Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*), we have $f \in \mathcal{M}_+$. Then, from Lemma 789 (*integral in $\mathcal{M}_+$*), the integral of f is well-defined. Moreover, from Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), and **completeness of the extended real numbers**, the sequence $(\int f_n \, d\mu)_{n \in \mathbb{N}}$ is nondecreasing, hence convergent in $\overline{\mathbb{R}}$ towards

$$\lim_{n \to \infty} \int f_n \, d\mu = \sup_{n \in \mathbb{N}} \int f_n \, d\mu.$$

Thus, from Definition 2 (*supremum*, upper bound), and Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), for all $n \in \mathbb{N}$, we have

$$\int f_n \, d\mu \leqslant \int \sup_{n\in\mathbb{N}} f_n \, d\mu = \int f \, d\mu.$$

Hence, from Definition 2 (*supremum*, least upper bound), we have

$$\lim_{n\to\infty} \int f_n \, d\mu = \sup_{n\in\mathbb{N}} \int f_n \, d\mu \leqslant \int f \, d\mu.$$

Let $a \in (0,1)$. Let $\varphi \in \mathcal{SF}_+$. Assume that $\varphi \leqslant f$. Let $n \in \mathbb{N}$. Let $A_n \stackrel{\text{def.}}{=} \{a\varphi \leqslant f_n\} \subset X$. Then, from Definition 748 (*$\mathcal{SF}$, vector space of simple functions*, $-a\varphi \in \mathcal{SF}$), Lemma 759 (*$\mathcal{SF}$ is measurable*, $-a\varphi \in \mathcal{M}$), and Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*, $-a\varphi$ takes finite values), we have $f_n - a\varphi \in \mathcal{M}$. Thus, from Lemma 578 (*measurability of numeric function*, with $f_n - a\varphi$), and Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*), we have $A_n \in \Sigma$, and $\mathbb{1}_{A_n} \in \mathcal{M}_+$. Moreover, from **monotonicity of addition in $\overline{\mathbb{R}}_+$**, and **transitivity of order**, the sequence $(A_n)_{n\in\mathbb{N}}$ is nondecreasing.
Let $x \in X$. **Case $\varphi(x) = 0$.** Then, for all $n \in \mathbb{N}$, $x \in A_n$. **Case $\varphi(x) > 0$.** Then, since $0 < a < 1$, we have $a\varphi(x) < \varphi(x) \leqslant f(x)$. Thus, from **monotonicity of the limit**, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $a\varphi(x) \leqslant f_n(x)$, i.e. $x \in A_N$. Hence, we have $X = \bigcup_{n\in\mathbb{N}} A_n$.

Let $B \in \Sigma$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection (with $\mathrm{card}(I) = 2$)), **monotonicity of intersection**, and **distributivity of intersection over union**, the sequence $(B \cap A_n)_{n\in\mathbb{N}}$ belongs to Σ, is nondecreasing, and $\bigcup_{n\in\mathbb{N}}(B \cap A_n) = B$. Thus, from Lemma 617 (*measure is continuous from below*), and Definition 616 (*continuity from below*), we have $\mu(B) = \mu\left(\bigcup_{n\in\mathbb{N}}(B\cap A_n)\right) = \lim_{n\to\infty}\mu(B\cap A_n)$.

Then, from Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*), there exists $k \in \mathbb{N}$, $(a_i)_{i\in[0..k]} \in \mathbb{R}_+$, and $(B_i)_{i\in[0..k]} \in \Sigma$ such that

$$\varphi = \sum_{i\in[0..k]} a_i \mathbb{1}_{B_i} \quad \text{and} \quad \int \varphi \, d\mu = \sum_{i\in[0..k]} a_i \mu(B_i).$$

Thus, from Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*), we have

$$\forall n \in \mathbb{N}, \int \varphi \mathbb{1}_{A_n} \, d\mu = \int \sum_{i\in[0..k]} a_i \mathbb{1}_{B_i\cap A_n} \, d\mu = \sum_{i\in[0..k]} a_i \mu(B_i \cap A_n).$$

Hence, from the previous result on the measure μ, **linearity of the limit**, and Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*, with $a > 0$), we have

$$\int \varphi \, d\mu = \sum_{i\in[0..k]} a_i \lim_{n\to\infty} \mu(B_i \cap A_n) = \lim_{n\to\infty} \int \varphi \mathbb{1}_{A_n} \, d\mu = \frac{1}{a} \lim_{n\to\infty} \int a\varphi \mathbb{1}_{A_n} \, d\mu.$$

Let $n \in \mathbb{N}$. Then, from the definition of the A_n's, and **the definition of the indicator function**, we have $a\varphi \mathbb{1}_{A_n} \leqslant f_n \mathbb{1}_{A_n} \leqslant f_n$. Thus, from **compatibility of the multiplication by a positive number with order**, Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), and **monotonicity and linearity of the limit**, we have

$$\int \varphi \, d\mu \leqslant \frac{1}{a} \lim_{n\to\infty} \int f_n \, d\mu.$$

Then, from **left continuity of the division at $a = 1$**, and **monotonicity of the limit**,

$$\forall \varphi \in \mathcal{SF}_+, \ \varphi \leqslant f \Rightarrow \int \varphi \, d\mu \leqslant \lim_{n\to\infty} \int f_n \, d\mu.$$

Hence, from Lemma 789 (*integral in $\mathcal{M}_+$*), and Definition 2 (*supremum*, least upper bound),

$$\int f\,d\mu = \sup_{\substack{\varphi\in\mathcal{SF}_+\\ \varphi\leqslant f}} \int \varphi\,d\mu \leqslant \lim_{n\to\infty} \int f_n\,d\mu.$$

Therefore, we have $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$. □

Lemma 797 ***(integral in $\mathcal{M}_+$ is homogeneous at ∞).***
Let (X,Σ,μ) be a measure space. Let $f\in\mathcal{M}_+$. Then, $\infty f\in\mathcal{M}_+$, and we have

$$\int \infty f\,d\mu = \infty \int f\,d\mu. \tag{13.56}$$

Proof. From Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*, $f\geqslant 0$), **ordered set properties of $\overline{\mathbb{R}}_+$**, Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*), and Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*), $(nf)_{n\in\mathbb{N}}$ is a nondecreasing sequence in $\mathcal{M}_+$ such that $\lim_{n\to\infty} nf = \infty f\in\mathcal{M}_+$. Therefore, from Theorem 796 (*Beppo Levi, monotone convergence*), Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*), and **positive homogeneity of the limit in $\overline{\mathbb{R}}_+$**, we have

$$\begin{aligned}\int \infty f\,d\mu &= \int \left(\lim_{n\to\infty} n\right) f\,d\mu = \int \left(\lim_{n\to\infty} nf\right)\,d\mu\\ &= \lim_{n\to\infty}\left(\int nf\,d\mu\right) = \lim_{n\to\infty}\left(n\int f\,d\mu\right) = \left(\lim_{n\to\infty} n\right)\int f\,d\mu = \infty\int f\,d\mu.\end{aligned}$$

□

Definition 798 ***(adapted sequence).***
Let (X,Σ,μ) be a measure space. Let $f\in\mathcal{M}_+$. Let $(\varphi_n)_{n\in\mathbb{N}}\in\mathcal{SF}_+$. The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is called *adapted sequence for f* iff it is nondecreasing and $f=\lim_{n\in\mathbb{N}}\varphi_n=\sup_{n\in\mathbb{N}}\varphi_n$.

Lemma 799 ***(adapted sequence in $\mathcal{M}_+$).***
Let (X,Σ,μ) be a measure space. Let $f\in\mathcal{M}_+$. For all $n\in\mathbb{N}$, let $\varphi_n : X\to\mathbb{R}_+$ defined by

$$\forall x\in X,\quad \varphi_n(x) \stackrel{\text{def.}}{=} \begin{cases} \dfrac{\lfloor 2^n f(x)\rfloor}{2^n} & \textit{when } f(x)<n,\\ n & \textit{otherwise.}\end{cases} \tag{13.57}$$

Then, $(\varphi_n)_{n\in\mathbb{N}}$ is an adapted sequence for f.

Proof. Let $n\in\mathbb{N}$. Let $x\in X$. Assume that $f(x)<n$. Let $i\stackrel{\text{def.}}{=}\lfloor 2^n f(x)\rfloor\in\mathbb{N}$. Then, from **the definition of the floor function**, we have $0\leqslant i\leqslant 2^n f(x)<i+1\leqslant n2^n$. Thus, for all $i\in\mathbb{N}$ such that $0\leqslant i<n2^n$, for all $x\in X$ such that $\frac{i}{2^n}\leqslant f(x)<\frac{i+1}{2^n}$, we have $\varphi_n(x)=\frac{i}{2^n}$. Hence,

$$\varphi_n = \sum_{0\leqslant i<n2^n} \frac{i}{2^n}\mathbb{1}_{\left\{\frac{i}{2^n}\leqslant f<\frac{i+1}{2^n}\right\}} + n\,\mathbb{1}_{\{f\geqslant n\}}.$$

Moreover, from Lemma 578 (*measurability of numeric function*), and Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection (with $\mathrm{card}(I)=2$)), we have

$$\left\{\frac{i}{2^n}\leqslant f<\frac{i+1}{2^n}\right\} = \left\{\frac{i}{2^n}\leqslant f\right\}\cap\left\{f<\frac{i+1}{2^n}\right\}\in\Sigma \quad\text{and}\quad \{f\geqslant n\}\in\Sigma.$$

Hence, from Definition 748 (*$\mathcal{SF}$, vector space of simple functions*), and Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*), we have $\varphi_n\in\mathcal{SF}_+$.

Let $n \in \mathbb{N}$. Let $x \in X$. **Case $n+1 \leqslant f(x)$.** Then, we have

$$\varphi_n(x) = n < n+1 = \varphi_{n+1}(x).$$

Case $f(x) < n+1$. Let $i \stackrel{\text{def.}}{=} \lfloor 2^{n+1} f(x) \rfloor$. Then, we have

$$\varphi_{n+1}(x) = \frac{i}{2^{n+1}} \quad \text{with } 0 \leqslant \frac{i}{2^{n+1}} \leqslant f(x) < \frac{i+1}{2^{n+1}}.$$

Case $\frac{i}{2^{n+1}} \geqslant n$. Then, $f(x) \geqslant n$ and we have

$$\varphi_n(x) = n \leqslant \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

Case $\frac{i}{2^{n+1}} < n$ and i even. Let $j \stackrel{\text{def.}}{=} \frac{i}{2} \in \mathbb{N}$. Then, from **ordered field properties of $\mathbb{R}$**, we have

$$\frac{j}{2^n} \leqslant f(x) < \frac{j}{2^n} + \frac{1}{2^{n+1}} < \frac{j+1}{2^n}.$$

Thus, from **the definition of the floor function**, we have $j = \lfloor 2^n f(x) \rfloor$ and

$$\varphi_n(x) = \frac{j}{2^n} = \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

Case $\frac{i}{2^{n+1}} < n$ and i odd. Let $j \stackrel{\text{def.}}{=} \frac{i-1}{2} \in \mathbb{N}$. Then, from **ordered field properties of $\mathbb{R}$**, we have

$$\frac{j}{2^n} \leqslant \frac{j}{2^n} + \frac{1}{2^{n+1}} \leqslant f(x) < \frac{j+1}{2^n}.$$

Thus, from **the definition of the floor function**, we have $j = \lfloor 2^n f(x) \rfloor$ and

$$\varphi_n(x) = \frac{j}{2^n} = \frac{i-1}{2^{n+1}} < \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

Thus, we always have $\varphi_n(x) \leqslant \varphi_{n+1}(x)$. Hence, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is nondecreasing.

Let $x \in X$. We have $f(x) \in \overline{\mathbb{R}}_+$. **Case $f(x) = \infty$.** Then, for all $n \in \mathbb{N}$, we have $\varphi_n(x) = n$. Hence, $\lim_{n\to\infty} \varphi_n(x) = f(x)$. **Case $f(x) \in \mathbb{R}_+$.** Then, from **the Archimedean property of $\mathbb{R}$**, there exists $N \in \mathbb{N}$ such that $f(x) < N$. Let $n \in \mathbb{N}$ such that $N \leqslant n$. Then, from **the definition of the floor function**, we have

$$2^n \varphi_n(x) = \lfloor 2^n f(x) \rfloor \leqslant 2^n f(x) < \lfloor 2^n f(x) \rfloor + 1 = 2^n \varphi_n(x) + 1.$$

Thus, from **ordered field properties of $\mathbb{R}$ (with $2^n > 0$)**, we have $f(x) - \frac{1}{2^n} < \varphi_n(x) \leqslant f(x)$. Hence, from **the squeeze theorem**, we have $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

Therefore, from Definition 798 (*adapted sequence*), $(\varphi_n)_{n\in\mathbb{N}}$ is an adapted sequence for f. □

Lemma 800 *(usage of adapted sequences).* *Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}_+$. Let $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{SF}_+$ be an adapted sequence of f. Then, we have*

$$\int f \, d\mu = \int \lim_{n\to\infty} \varphi_n \, d\mu = \lim_{n\to\infty} \int \varphi_n \, d\mu. \tag{13.58}$$

Proof. Direct consequence of Definition 798 (*adapted sequence*), Lemma 799 (*adapted sequence in $\mathcal{M}_+$*), Theorem 796 (*Beppo Levi, monotone convergence*). and Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*). □

Lemma 801 *(integral in $\mathcal{M}_+$ is additive).*
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}_+$. Then, $f + g \in \mathcal{M}_+$, and we have

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu. \tag{13.59}$$

Proof. From Lemma 597 (*$\mathcal{M}_+$ is closed under addition*), we have $f+g \in \mathcal{M}_+$.

From Lemma 799 (*adapted sequence in $\mathcal{M}_+$*), let $(\varphi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}} \in \mathcal{SF}_+$ be adapted sequences for f and g. Then, from Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*, additivity), **monotonicity of addition**, and **additivity of the limit**, $(\varphi_n + \psi_n)_{n\in\mathbb{N}} \in \mathcal{SF}_+$ is an adapted sequence for $f+g$. Let $n \in \mathbb{N}$. Then, from Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*, additivity), we have

$$\int (\varphi_n + \psi_n)\, d\mu = \int \varphi_n \, d\mu + \int \psi_n \, d\mu.$$

Therefore, from **linearity of the limit when n goes to infinity**, and Lemma 800 (*usage of adapted sequences*), we have

$$\int (f+g)\, d\mu = \int f \, d\mu + \int g \, d\mu.$$

□

Lemma 802 ***(integral in $\mathcal{M}_+$ is positive linear).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}_+$. Let $a \in \overline{\mathbb{R}}_+$. Then, $f+g, af \in \mathcal{M}_+$, and we have*

$$\int (f+g)\, d\mu = \int f \, d\mu + \int g \, d\mu \quad \text{and} \quad \int af \, d\mu = a \int f \, d\mu. \tag{13.60}$$

Proof. Direct consequence of Lemma 801 (*integral in $\mathcal{M}_+$ is additive*), Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*), and Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*). □

Lemma 803 ***(integral in $\mathcal{M}_+$ is σ-additive).***
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Then, $\sum_{n\in\mathbb{N}} f_n \in \mathcal{M}_+$, and we have

$$\int \left(\sum_{n\in\mathbb{N}} f_n \right) d\mu = \sum_{n\in\mathbb{N}} \left(\int f_n \, d\mu \right). \tag{13.61}$$

Proof. Direct consequence of Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*), and Theorem 796 (*Beppo Levi, monotone convergence*, with nondecreasing sequence $(\sum_{i\in[0,n]} f_i)_{n\in\mathbb{N}}$). □

Lemma 804 ***(integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts).*** *Let (X, Σ) be a measurable space. Let $f \in \mathcal{M}$. Then, we have*

$$\int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu. \tag{13.62}$$

Proof. Direct consequence of Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f|$ belongs to $\mathcal{M}_+$), Lemma 594 (*measurability of nonnegative and nonpositive parts*, $f^+, f^- \in \mathcal{M}_+$), Lemma 403 (*decomposition into nonnegative and nonpositive parts*, $|f| = f^+ + f^-$), and Lemma 801 (*integral in $\mathcal{M}_+$ is additive*). □

Lemma 805 ***(compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts).*** *Let (X, Σ) be a measurable space. Let $f, g \in \mathcal{M}$ such that $f+g \in \mathcal{M}$. Then, we have*

$$\int (f+g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int (f+g)^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu. \tag{13.63}$$

Proof. Direct consequence of Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*), and Lemma 801 (*integral in $\mathcal{M}_+$ is additive*). □

Lemma 806 ***(integral in $\mathcal{M}_+$ is almost definite).***
Let (X, Σ, μ) be a measure space. Let f in $\mathcal{M}_+$. Then, we have

$$\int f\, d\mu = 0 \iff f \overset{\mu\,\text{a.e.}}{=} 0. \tag{13.64}$$

Proof. Let $A \overset{\text{def.}}{=} \{f > 0\} = f^{-1}(0, \infty]$. Then, from Lemma 578 (*measurability of numeric function*, $\underline{A \in \Sigma}$), Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, $\underline{\mathbb{1}_A \in \mathcal{M}_+}$), Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*, $\underline{\infty f, \infty \mathbb{1}_A \in \mathcal{M}_+}$), Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*), Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*, $\underline{\infty f = \infty \mathbb{1}_A}$), and Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*) we have

$$\infty \int f\, d\mu = \int \infty f\, d\mu = \int \infty \mathbb{1}_A\, d\mu = \infty \int \mathbb{1}_A\, d\mu = \infty \mu(A).$$

Therefore, from Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*, multiplication by ∞ is definite in $\overline{\mathbb{R}}_+$), Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*, $\underline{\{f = 0\}^c = A}$), Lemma 636 (*negligibility of measurable subset*), and Definition 641 (*property almost satisfied*), we have

$$\begin{aligned} \int f\, d\mu = 0 \quad &\iff \quad \infty \int f\, d\mu = 0 \quad \iff \quad \infty\, \mu(A) = 0 \\ &\iff \quad \mu(A) = 0 \quad \iff \quad \mu(\{f = 0\}^c) = 0 \quad \iff \quad f \overset{\mu\,\text{a.e.}}{=} 0. \end{aligned}$$

□

Lemma 807 ***(compatibility of integral in $\mathcal{M}_+$ with almost binary relation).***
Let (X, Σ, μ) be a measure space. Let $\mathcal{R}$ be a binary relation on $\mathcal{M}_+$. Let $\mathcal{R}'$ be a binary relation on $\overline{\mathbb{R}}_+$. Assume that we have the properties $0\, \mathcal{R}\, 0$ and

$$\forall f, g \in \mathcal{M}_+, \quad f\, \mathcal{R}\, g \implies \int f\, d\mu\, \mathcal{R}' \int g\, d\mu. \tag{13.65}$$

Then, we have the property

$$\forall f, g \in \mathcal{M}_+, \quad f\, \mathcal{R}_{\mu\,\text{a.e.}}\, g \implies \int f\, d\mu\, \mathcal{R}' \int g\, d\mu. \tag{13.66}$$

Proof. Let $f, g \in \mathcal{M}_+$. Assume that $f\, \mathcal{R}_{\mu\,\text{a.e.}}\, g$ holds.

From Definition 650 (*almost binary relation*), Definition 641 (*property almost satisfied*), Definition 631 (*negligible subset*), and **monotonicity of complement**, let $A \in \Sigma$ such that $\mu(A) = 0$ and $A^c \subset \{f\, \mathcal{R}\, g\}$, i.e. such that $f_{|A^c}\, \mathcal{R}\, g_{|A^c}$ holds. Thus, from **the definition of the indicator function**, and since $0\, \mathcal{R}\, 0$, we have $f\, \mathbb{1}_{A^c}\, \mathcal{R}\, g\, \mathbb{1}_{A^c}$. Then, from Definition 611 (*measure*), Definition 516 (*measurable space*, $\underline{\Sigma \text{ is a } \sigma\text{-algebra}}$), Definition 474 (*$\sigma$-algebra*, $\underline{A^c \in \Sigma}$), Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, $\underline{\mathbb{1}_A, \mathbb{1}_{A^c} \in \mathcal{M}_+}$), and Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*), we have $f\, \mathbb{1}_A, f\, \mathbb{1}_{A^c}, g\, \mathbb{1}_A, g\, \mathbb{1}_{A^c} \in \mathcal{M}_+$. Thus, from assumption, the property $\int f\, \mathbb{1}_{A^c}\, d\mu\, \mathcal{R}' \int g\, \mathbb{1}_{A^c}\, d\mu$ holds.

Let $h \in \{f, g\}$. Then, from **the definition of the indicator function ($\{\mathbb{1}_A = 0\}^c$ is A)**, Lemma 636 (*negligibility of measurable subset*, $\underline{\{\mathbb{1}_A = 0\}^c \in \mathbf{N}}$), Definition 641 (*property almost satisfied*, $\underline{\mathbb{1}_A \overset{\mu\,\text{a.e.}}{=} 0}$), Lemma 660 (*compatibility of almost equality with operator*, with the unary operator left multiplication by h), and Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), we have $h\, \mathbb{1}_A \overset{\mu\,\text{a.e.}}{=} h0 = 0$. Thus, from Lemma 643 (*everywhere implies almost everywhere*), and Lemma 657 (*almost equality is equivalence relation*, transitivity), we have $h\, \mathbb{1}_A \overset{\mu\,\text{a.e.}}{=} 0$. Hence, from **properties of the indicator function**, Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*), Lemma 801 (*integral in $\mathcal{M}_+$ is additive*), Lemma 806 (*integral* *in $\mathcal{M}_+$ is almost definite*), and Lemma 283 (*zero is identity element for addition in $\overline{\mathbb{R}}$*), we have for all $h \in \{f, g\}$,

$$\int h\,d\mu = \int h(\mathbb{1}_A + \mathbb{1}_{A^c})\,d\mu = \int h\,\mathbb{1}_A\,d\mu + \int h\,\mathbb{1}_{A^c}\,d\mu = \int h\,\mathbb{1}_{A^c}\,d\mu.$$

Therefore, $\int f\,d\mu\,\mathcal{R}'\,\int g\,d\mu$ also holds. □

Lemma 808 ***(compatibility of integral in $\mathcal{M}_+$ with almost equality).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}_+$. Then, we have

$$f \overset{\mu\text{ a.e.}}{=} g \quad\Longrightarrow\quad \int f\,d\mu = \int g\,d\mu. \tag{13.67}$$

Proof. Direct consequence of Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*, with $\mathcal{R} = \mathcal{R}' \overset{\text{def.}}{=}$ equality), and Lemma 789 (*integral in $\mathcal{M}_+$*, integral is a function). □

Lemma 809 ***(integral in $\mathcal{M}_+$ is almost monotone).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}_+$. Then, we have

$$f \overset{\mu\text{ a.e.}}{\leqslant} g \quad\Longrightarrow\quad \int f\,d\mu \leqslant \int g\,d\mu. \tag{13.68}$$

Proof. Direct consequence of Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*, with $\mathcal{R} = \mathcal{R}' \overset{\text{def.}}{=}$ inequality), and Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*). □

Lemma 810 ***(Bienaymé–Chebyshev inequality).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Let $a \in \overline{\mathbb{R}}_+^\star$. Then, we have

$$a\mu\left(\{|f| \geqslant a\}\right) \leqslant \int |f|\,d\mu. \tag{13.69}$$

Proof. Let $A \overset{\text{def.}}{=} \{|f| \geqslant a\}$. Then, from Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f|$ belongs to $\mathcal{M}_+ \subset \mathcal{M}$), and Lemma 578 (*measurability of numeric function*), we have $A \in \Sigma$. Moreover, from **the definition of the indicator function**, and Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*), we have $|f| \geqslant a\,\mathbb{1}_A$. Therefore, from Lemma 569 (*measurability of indicator function*), Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*), Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*), Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*), Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), and Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*), we have $a\,\mathbb{1}_A \in \mathcal{M}_+$ and

$$\int |f|\,d\mu \geqslant \int a\,\mathbb{1}_A\,d\mu = a \int \mathbb{1}_A\,d\mu = a\mu(A).$$

□

Lemma 811 ***(integrable in $\mathcal{M}_+$ is almost finite).*** *Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}_+$ be μ-integrable in $\mathcal{M}_+$. Then, we have $\mu(f^{-1}(\infty)) = 0$, i.e. $f \overset{\mu\text{ a.e.}}{<} \infty$.*

Proof. Direct consequence of Lemma 789 (*integral in $\mathcal{M}_+$*, $f \in \mathcal{M}_+$), Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), Lemma 810 (*Bienaymé–Chebyshev inequality*, with $a \overset{\text{def.}}{=} \infty$), Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*, $|f| = f$), Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*, $A \overset{\text{def.}}{=} \{|f| \geqslant \infty\} = f^{-1}(\infty)$), Lemma 789 (*integral in $\mathcal{M}_+$*, $\int |f|\,d\mu < \infty$), **compatibility of multiplication with order in $\overline{\mathbb{R}}$ ($\mu(A) \leqslant 0$)**, Definition 611 (*measure*, nonnegativeness, i.e. $\mu(A) = 0$), Lemma 636 (*negligibility of measurable subset*, $A \in \mathbf{N}$), and Definition 641 (*property almost satisfied*, $A^c = \{f < \infty\}$), □

Lemma 812 ***(bounded by integrable in $\mathcal{M}_+$ is integrable).***
Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}_+$. Then, f is μ-integrable in $\mathcal{M}_+$ iff there exists $g : X \to \overline{\mathbb{R}}_+$ such that g is μ-integrable in $\mathcal{M}_+$ and $f \leqslant g$.

Proof. **"Left" implies "right".** Direct consequence of Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, reflexivity, with $g \stackrel{\text{def.}}{=} f$).

"Right" implies "left". Direct consequence of Lemma 789 (*integral in $\mathcal{M}_+$*, μ-integrability), Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), and Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, transitivity).

Therefore, we have the equivalence. □

Lemma 813 ***(integral in $\mathcal{M}_+$ over subset).*** *Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, we have $f_{|A} \in \mathcal{M}_+(A, \Sigma \overline{\cap} A)$ iff $\hat{f}\,\mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$. If so, we have*

$$\int f_{|A}\, d\mu_A = \int \hat{f}\,\mathbb{1}_A\, d\mu. \tag{13.70}$$

This integral is still denoted $\int_A f\, d\mu$; it is still called integral of f over A.

Proof. **Equivalence.** Direct consequence of Lemma 592 (*measurability of restriction*), Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), and **nonnegativeness of the indicator function**.

Identity. Direct consequence of Lemma 799 (*adapted sequence in $\mathcal{M}_+$*, let $(\hat{\varphi}_n)_{n\in\mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$ be an adapted sequence for $\hat{f}\,\mathbb{1}_A$), Lemma 761 (*$\mathcal{SF}$ is closed under restriction*, $(\hat{\varphi}_n)_{|A}$ belongs to $\mathcal{SF}_+(A, \Sigma \overline{\cap} A)$), **compatibility of restriction of function with monotonicity and limit**, Definition 798 (*adapted sequence*, $((\hat{\varphi}_n)_{|A})_{n\in\mathbb{N}}$ is an adapted sequences for $f_{|A}$), Lemma 800 (*usage of adapted sequences*), and Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*). □

Lemma 814 ***(integral in $\mathcal{M}_+$ over subset is σ-additive).*** *Let (X, Σ, μ) be a measure space. Let $I \subset \mathbb{N}$. Let $A, (A_i)_{i\in I} \in \Sigma$. Assume that $(A_i)_{i\in I}$ is a pseudopartition of A. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$.*
Then, $\hat{f}\,\mathbb{1}_A = \sum_{i\in I} \hat{f}\,\mathbb{1}_{A_i}$, and $\hat{f}\,\mathbb{1}_A \in \mathcal{M}_+$ iff for all $i \in I$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{M}_+$. If so, we have

$$\int_A f\, d\mu = \sum_{i\in I} \int_{A_i} f\, d\mu. \tag{13.71}$$

Proof. **(1). $\hat{f}\,\mathbb{1}_A = \sum_{i\in I} \hat{f}\,\mathbb{1}_{A_i}$.** Direct consequence of Lemma 735 (*$\mathcal{IF}$ is σ-additive*, with $I \stackrel{\text{def.}}{=} \mathbb{N}$), and **left distributivity of multiplication over countable addition in $\overline{\mathbb{R}}_+$**.

"Left" implies "right". Direct consequence of Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*, $\mathbb{1}_{A_i} = \mathbb{1}_{A\cap A_i} = \mathbb{1}_A\,\mathbb{1}_{A_i}$), **associativity of multiplication in $\overline{\mathbb{R}}$**, Lemma 569 (*measurability of indicator function*), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$), **nonnegativeness of the indicator function**, Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), and Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*).

"Right" implies "left". Direct consequence of (1), and Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*).

Therefore, we have the equivalence.

Identity. Direct consequence of Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, with A, then A_i), (1), and Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*). □

Lemma 815 ***(integral in $\mathcal{M}_+$ over singleton).*** *Let (X, Σ, μ) be a measure space. Let $a \in X$. Assume that $\{a\} \in \Sigma$. Let $f : X \to \overline{\mathbb{R}}_+$. Then, we have $f\,\mathbb{1}_{\{a\}} = f(a)\,\mathbb{1}_{\{a\}} \in \mathcal{M}_+$, and*

$$\int_{\{a\}} f\,d\mu = f(a)\mu(\{a\}). \tag{13.72}$$

Proof. From **the definition of the indicator function**, Lemma 569 (*measurability of indicator function*), and Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*), we have $f\,\mathbb{1}_{\{a\}} = f(a)\,\mathbb{1}_{\{a\}} \in \mathcal{M}_+$. Then, from Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, with $A \stackrel{\text{def.}}{=} \{a\}$, $Y \stackrel{\text{def.}}{=} X$), Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*, homogeneity in $\overline{\mathbb{R}}_+$), and Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*), we have

$$\int_{\{a\}} f\,d\mu = \int f\,\mathbb{1}_{\{a\}}\,d\mu = f(a)\int \mathbb{1}_{\{a\}}\,d\mu = f(a)\mu(\{a\}).$$

□

Remark 816. See the sketch of next proof in Section 5.4.

Theorem 817 ***(Fatou's lemma).***
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Then, $\liminf_{n\to\infty} f_n \in \mathcal{M}_+$, and we have

$$\int \liminf_{n\to\infty} f_n\,d\mu \leqslant \liminf_{n\to\infty} \int f_n\,d\mu. \tag{13.73}$$

Proof. From Lemma 392 (*limit inferior bounded from below*, with $m \stackrel{\text{def.}}{=} 0$), and Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*), we have $\liminf_{n\to\infty} f_n \in \mathcal{M}_+$.

Let $n \in \mathbb{N}$. Let $g_n \stackrel{\text{def.}}{=} \inf_{p\in\mathbb{N}} f_{n+p}$. Then, from **monotonicity of infimum**, Lemma 376 (*infimum of bounded sequence is bounded*, with $a \stackrel{\text{def.}}{=} 0$), and Lemma 586 (*$\mathcal{M}$ is closed under infimum*), $(g_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence in $\mathcal{M}_+$. Thus, from Definition 390 (*pointwise convergence*), **properties of nondecreasing sequences in the ordered set $\overline{\mathbb{R}}$**, Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*), $(g_n)_{n\in\mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}_+$ towards the measurable function $g \stackrel{\text{def.}}{=} \lim_{n\in\mathbb{N}} g_n$. Moreover, from **the nondecreasing property of the sequence in the ordered set $\overline{\mathbb{R}}$**, and Lemma 378 (*limit inferior*), we have

$$g = \sup_{n\in\mathbb{N}} g_n = \liminf_{n\to\infty} f_n.$$

Hence, from Theorem 796 (*Beppo Levi, monotone convergence*, with $(g_n)_{n\in\mathbb{N}}$), g is measurable and nonnegative, and we have

$$\int \liminf_{n\to\infty} f_n\,d\mu = \int g\,d\mu = \lim_{n\to\infty} \int g_n\,d\mu.$$

Let $n, p \in \mathbb{N}$. Let $x \in X$. Then, from Definition 9 (*infimum*, lower bound), we have

$$g_n(x) \leqslant f_{n+p}(x).$$

Then, from Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), we have

$$\int g_n\,d\mu \leqslant \int f_{n+p}\,d\mu.$$

Hence, from Definition 9 (*infimum*, greatest lower bound), we have

$$\int g_n\,d\mu \leqslant \inf_{p\in\mathbb{N}} \int f_{n+p}\,d\mu.$$

Therefore, from **monotonicity of the limit in the ordered set $\overline{\mathbb{R}}$**, and Lemma 378 (*limit inferior*), we have

$$\int \liminf_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \leqslant \liminf_{n\to\infty} \int f_n \, d\mu.$$

□

Lemma 818 ***(integral in $\mathcal{M}_+$ of pointwise convergent sequence).***
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}_+$. Assume that the sequence is pointwise convergent towards f such that for all $n \in \mathbb{N}$, $f_n \leqslant f$. Then, $f \in \mathcal{M}_+$, and we have

$$\int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu. \tag{13.74}$$

Proof. From Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*), we have

$$f = \lim_{n\to\infty} f_n \in \mathcal{M}_+.$$

Moreover, from Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), we have

$$\forall n \in \mathbb{N}, \quad \int f_n \, d\mu \leqslant \int f \, d\mu.$$

Thus, from Definition 2 (*supremum*, least upper bound), Lemma 391 (*limit inferior and limit superior of pointwise convergent*), and Theorem 817 (*Fatou's lemma*), we have

$$\limsup_{n\to\infty} \int f_n \, d\mu \leqslant \int f \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu \leqslant \liminf_{n\to\infty} \int f_n \, d\mu$$

Therefore, from Lemma 396 (*limit inferior, limit superior and pointwise convergence*), we have

$$\liminf_{n\to\infty} \int f_n \, d\mu = \limsup_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

□

Lemma 819 ***(integral in $\mathcal{M}_+$ for counting measure).***
Let (X, Σ) be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{M}_+$. Then, we have

$$\int f \, d\delta_Y = \sum_{y\in Y} f(y). \tag{13.75}$$

Proof. Direct consequence of Lemma 800 (*usage of adapted sequences*), Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*), and **compatibility of (possibly uncountable) addition in $\overline{\mathbb{R}}_+$ with limit**. □

Lemma 820 ***(integral in $\mathcal{M}_+$ for counting measure on $\mathbb{N}$).***
Let $f : \mathbb{N} \to \overline{\mathbb{R}}_+$ be a nonnegative sequence. Then, we have

$$\int f \, d\delta_{\mathbb{N}} = \sum_{n\in\mathbb{N}} f(n). \tag{13.76}$$

Proof. Direct consequence of Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*, with $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$ and $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$). □

Remark 821. Note that the previous lemma makes nonnegative series be Lebesgue integrals for the counting measure on natural numbers. Thus, the theory of nonnegative series can be derived from Lebesgue integration for nonnegative measurable functions.

Lemma 822 ***(integral in $\mathcal{M}_+$ for Dirac measure).***
Let (X, Σ) be a measurable space. Let $\{a\} \in \Sigma$. Let $f \in \mathcal{M}_+$. Then, we have

$$\int f \, d\delta_a = f(a). \tag{13.77}$$

Proof. Direct consequence of Definition 675 (*Dirac measure*), and Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*, with $Y \stackrel{\text{def.}}{=} \{a\}$). □

13.4 Measure and integration over product space

13.4.1 Measure over product space

Remark 823. For the sake of simplicity, we only present this section in the case of the product of two measure spaces. When $i \in \{1,2\}$, the complement $\{1,2\} \setminus \{i\}$ is $\{3-i\}$.

The definition of sections and some of their properties were presented in Section 9.4.

Lemma 824 ***(measure of section).*** *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $A \in \Sigma_1 \otimes \Sigma_2$. Let $i \in \{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Then, the function F_i^A defined by*

$$\forall x_i \in X_i, \quad F_i^A(x_i) \stackrel{\text{def.}}{=} \mu_j(s_i(x_i, A)) \tag{13.78}$$

is well-defined on X_i, and takes its values in $\overline{\mathbb{R}}_+$. It is called measure of i-th section of A.

Proof. Direct consequence of Lemma 551 (*measurability of section*), and Definition 611 (*measure*, nonnegativeness). □

Lemma 825 ***(measure of section of product).*** *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. Let $i \in \{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Then, we have*

$$F_i^{A_1 \times A_2} = \mu_j(A_j)\, \mathbb{1}_{A_i}. \tag{13.79}$$

Proof. Direct consequence of Lemma 542 (*product of measurable subsets is measurable*, $A_1 \times A_2$ belongs to Σ), Lemma 824 (*measure of section*), Lemma 549 (*section of product*), Definition 611 (*measure*, $\mu_j(\varnothing) = 0$ and $\mu_j(A_j) \in \overline{\mathbb{R}}_+$), Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), and **the definition of the indicator function**. □

Remark 826. The next proof follows the monotone class theorem scheme (see Section 4.3).

Lemma 827 ***(measurability of measure of section (finite)).*** *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $i \in \{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Assume that μ_j is finite. Let $A \in \Sigma_1 \otimes \Sigma_2$. Then, we have $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$.*

Proof. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$. From Definition 217 (*product of subsets of parties*), Definition 541 (*tensor product of σ-algebras*), and Definition 442 (*generated set algebra*), let $\overline{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$, $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$, and $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\overline{\Sigma})$. Let $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \in \Sigma \,|\, F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)\}$.

(1). $\overline{\Sigma} \subset \mathcal{S}_i$. Let $A \in \overline{\Sigma}$. Let $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$ such that $A = A_1 \times A_2$.

From Lemma 825 (*measure of section of product*, $F_i^A = \mu_j(A_j)\, \mathbb{1}_{A_i}$), Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, $\mathbb{1}_{A_i} \in \mathcal{M}_+(X_i, \Sigma_i)$), Definition 611 (*measure*, $\mu_j(A_j) \in \overline{\mathbb{R}}_+$), and Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*, with $a \stackrel{\text{def.}}{=} \mu_j(A_j)$), F_i^A belongs to $\mathcal{M}_+(X_i, \Sigma_i)$. Thus, from Lemma 542 (*product of measurable subsets is measurable*, $A \in \Sigma$), and the definition of $\mathcal{S}_i$, we have $A \in \mathcal{S}_i$. Hence, we have $\overline{\Sigma} \subset \mathcal{S}_i$.

(2). $\mathcal{A} \subset \mathcal{S}_i$. Let $A \in \mathcal{A}$.

Then, from Lemma 443 (*generated set algebra is minimum*), Definition 541 (*tensor product of σ-algebras*, $\Sigma = \Sigma_X(\overline{\Sigma})$), Lemma 490 (*$\sigma$-algebra contains set algebra*, with $G \stackrel{\text{def.}}{=} \overline{\Sigma}$, thus $\mathcal{A} \subset \Sigma$), Lemma 505 (*set algebra generated by product of σ-algebras*), and Lemma 542 (*product of measurable subsets is measurable*, $\overline{\Sigma} \subset \Sigma$), we have $A \in \Sigma$, and there exists $n \in \mathbb{N}$, and $(A_p)_{p \in [0..n]}$ in $\overline{\Sigma} \subset \Sigma$ such that, for all $p, q \in [0..n]$, $p \neq q \Rightarrow A_p \cap A_q = \varnothing$, and $A = \biguplus_{p \in [0..n]} A_p$.

Let $x_i \in X_i$. Then, from Lemma 824 (*measure of section*), Lemma 552 (*countable union of sections is measurable*, with $I \stackrel{\text{def.}}{=} [0..n]$), Lemma 550 (*compatibility of section with set operations*, the $s_i(x_i, A_p)$'s are pairwise disjoint), Definition 611 (*measure*, μ_j is σ-additive), and Definition 608 (*σ-additivity over measurable space*, with $I \stackrel{\text{def.}}{=} [0..n]$ and $\mu \stackrel{\text{def.}}{=} \mu_j$), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j\left(\biguplus_{p\in[0..n]} s_i(x_i, A_p)\right) = \sum_{p\in[0..n]} \mu_j(s_i(x_i, A_p)) = \sum_{p\in[0..n]} F_i^{A_p}(x_i).$$

Thus, from (1) ($A_p \in \mathcal{S}_i$), the definition of $\mathcal{S}_i$, and Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*, with $I \stackrel{\text{def.}}{=} [0..n]$, thus $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$), we have $A \in \mathcal{S}_i$. Hence, we have $\mathcal{A} \subset \mathcal{S}_i$.

(3). $\boldsymbol{\mathcal{S}_i}$ is monotone class. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}_i \subset \Sigma$.

Let $x_i \in X_i$. Let $n \in \mathbb{N}$. Then, from Lemma 551 (*measurability of section*), Lemma 552 (*countable union of sections is measurable*), and Lemma 553 (*countable intersection of sections is measurable*), we have

$$s_i(x_i, A_n), \quad s_i\left(x_i, \bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} s_i(x_i, A_n), \quad s_i\left(x_i, \bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} s_i(x_i, A_n) \quad \in \Sigma_j.$$

Assume first that, for all $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Let $A \stackrel{\text{def.}}{=} \bigcup_{n\in\mathbb{N}} A_n$.
Let $x_i \in X_i$. Then, from Lemma 824 (*measure of section*), the definition of A, Lemma 550 (*compatibility of section with set operations*, $(s_i(x_i, A_n))_{n\in\mathbb{N}}$ is nondecreasing), and Lemma 617 (*measure is continuous from below*, with μ_j), we have

$$\begin{aligned}F_i^A(x_i) = \mu_j(s_i(x_i, A)) &= \mu_j\left(s_i\left(x_i, \bigcup_{n\in\mathbb{N}} A_n\right)\right)\\ &= \mu_j\left(\bigcup_{n\in\mathbb{N}} s_i(x_i, A_n)\right) = \sup_{n\in\mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sup_{n\in\mathbb{N}} F_i^{A_n}(x_i).\end{aligned}$$

Thus, from the definition of $\mathcal{S}_i$, and Lemma 601 (*$\mathcal{M}_+$ is closed under supremum*), F_i^A belongs to $\mathcal{M}_+(X_i, \Sigma_i)$. Hence, from the definition of $\mathcal{S}_i$, we have $A \in \mathcal{S}_i$, i.e. $\mathcal{S}_i$ is closed under nondecreasing union.

Assume now that, for all $n \in \mathbb{N}$, $A_n \supset A_{n+1}$. Let $A \stackrel{\text{def.}}{=} \bigcap_{n\in\mathbb{N}} A_n$.
Let $x_i \in X_i$. Then, from Lemma 824 (*measure of section*), the definition of A, Lemma 550 (*compatibility of section with set operations*, $(s_i(x_i, A_n))_{n\in\mathbb{N}}$ is nonincreasing), Definition 622 (*finite measure*, with μ_j), Lemma 623 (*finite measure is bounded*, $\mu_j(s_i(x_i, A_0))$ is finite), and Lemma 619 (*measure is continuous from above*, with μ_j and $n_0 \stackrel{\text{def.}}{=} 0$), we have

$$\begin{aligned}F_i^A(x_i) = \mu_j(s_i(x_i, A)) &= \mu_j\left(s_i\left(x_i, \bigcap_{n\in\mathbb{N}} A_n\right)\right)\\ &= \mu_j\left(\bigcap_{n\in\mathbb{N}} s_i(x_i, A_n)\right) = \inf_{n\in\mathbb{N}} \mu_j(s_i(x_i, A_n)) = \inf_{n\in\mathbb{N}} F_i^{A_n}(x_i).\end{aligned}$$

Thus, from the definition of $\mathcal{S}_i$, and Lemma 600 (*$\mathcal{M}_+$ is closed under infimum*), $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$. Then, from the definition of $\mathcal{S}_i$, we have $A \in \mathcal{S}_i$, i.e. $\mathcal{S}_i$ is closed under nonincreasing intersection.

Hence, from Definition 448 (*monotone class*), $\mathcal{S}_i$ is a monotone class.

Therefore, from Definition 541 (*tensor product of σ-algebras*, $\Sigma = \Sigma_X(\overline{\Sigma})$), (2), (3), and Lemma 515 (*usage of monotone class theorem*, with $G \stackrel{\text{def.}}{=} \overline{\Sigma}$), we have for all $A \in \Sigma$, $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$. □

Lemma 828 *(measurability of measure of section).* *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $A \in \Sigma_1 \otimes \Sigma_2$. Let $i \in \{1, 2\}$. Then, we have $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$.*

Proof. Let $j \stackrel{\text{def.}}{=} 3-i$. From Lemma 625 (*equivalent definition of σ-finite measure*, with μ_j), there exists $(B_{j,n})_{n\in\mathbb{N}} \in \Sigma_j$ such that, for all $n \in \mathbb{N}$, $B_{j,n} \subset B_{j,n+1}$, $\mu_j(B_{j,n}) < \infty$, and $X_j = \bigcup_{n\in\mathbb{N}} B_{j,n}$.

For all $n \in \mathbb{N}$, for all $A \in \Sigma_1 \otimes \Sigma_2$, from Lemma 629 (*restricted measure*, with $Y \stackrel{\text{def.}}{=} B_{j,n}$), and Lemma 824 (*measure of section*, with $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$), let $\mu_{j,n}$ be the restricted measure, and $F_{i,n}^A$ be the measure of i-th section of A defined by

$$\mu_{j,n} \stackrel{\text{def.}}{=} (\mu_j)'_{B_{j,n}} = (A_j \in \Sigma_j \longmapsto \mu_j(A_j \cap B_{j,n})) \quad \text{and} \quad F_{i,n}^A \stackrel{\text{def.}}{=} (x_i \in X_i \longmapsto \mu_{j,n}(s_i(x_i, A))).$$

Let $n \in \mathbb{N}$. Then, from Definition 622 (*finite measure*, $\mu_{j,n}(X) = \mu_j(B_{j,n}) < \infty$), $\mu_{j,n}$ is a finite measure on (X_j, Σ_j). Moreover, from Lemma 827 (*measurability of measure of section (finite)*, with $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$ finite), we have $F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i)$.

Let $x_i \in X_i$. Then, from Lemma 551 (*measurability of section*), the definition of $B_{j,n}$, and **distributivity of intersection over union**, we have

$$\Sigma_j \ni s_i(x_i, A) = s_i(x_i, A) \cap \bigcup_{n\in\mathbb{N}} B_{j,n} = \bigcup_{n\in\mathbb{N}} (s_i(x_i, A) \cap B_{j,n}).$$

Thus, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection (with $\mathrm{card}(I) = 2$)), we have $s_i(x_i, A) \cap B_{j,n} \in \Sigma_j$. Hence, from Lemma 824 (*measure of section*), **monotonicity of intersection ($(s_i(x_i, A) \cap B_{j,n})_{n\in\mathbb{N}}$ is nondecreasing)**, Lemma 617 (*measure is continuous from below*, with μ_j and $A_n \stackrel{\text{def.}}{=} s_i(x_i, A) \cap B_{j,n}$), and the definition of $\mu_{j,n}$,

$$\begin{aligned} F_i^A(x_i) = \mu_j(s_i(x_i, A)) &= \mu_j\left(s_i(x_i, A) \cap \bigcup_{n\in\mathbb{N}} B_{j,n}\right) = \mu_j\left(\bigcup_{n\in\mathbb{N}} (s_i(x_i, A) \cap B_{j,n})\right) \\ &= \sup_{n\in\mathbb{N}} \mu_j(s_i(x_i, A) \cap B_{j,n}) = \sup_{n\in\mathbb{N}} \mu_{j,n}(s_i(x_i, A)) = \sup_{n\in\mathbb{N}} F_{i,n}^A(x_i). \end{aligned}$$

Therefore, from Lemma 601 (*$\mathcal{M}_+$ is closed under supremum*), we have

$$F_i^A = \sup_{n\in\mathbb{N}} F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i).$$

□

Definition 829 ***(tensor product measure).*** Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. A measure μ on the product measurable space (X, Σ) is called *tensor product measure (on (X, Σ) relying on μ_1 and μ_2)* iff

(13.80) $$\forall A_1 \in \Sigma_1,\ \forall A_2 \in \Sigma_2, \quad \mu(A_1 \times A_2) = \mu_1(A_1)\,\mu_2(A_2).$$

Definition 830 ***(candidate tensor product measure).*** Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $i \in \{1, 2\}$. The function defined by

(13.81) $$\forall A \in \Sigma_1 \otimes \Sigma_2, \quad (\mu_1 \otimes \mu_2)^i(A) \stackrel{\text{def.}}{=} \int F_i^A \, d\mu_i$$

is called *candidate tensor product measure*.

Lemma 831 ***(candidate tensor product measure is tensor product measure).*** *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $i \in \{1, 2\}$. Then, the candidate tensor product measure $(\mu_1 \otimes \mu_2)^i$ is a tensor product measure on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$.*

Proof. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$.

Measure. Let $j \stackrel{\text{def.}}{=} 3 - i$.

Let $A \in \Sigma$. Then, from Lemma 824 (*measure of section*), Definition 830 (*candidate tensor product measure*), Lemma 828 (*measurability of measure of section*, with the σ-finite measure μ_i), and Lemma 789 (*integral in $\mathcal{M}_+$*, nonnegativeness, with $X \stackrel{\text{def.}}{=} X_i$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_i$), the value $(\mu_1 \otimes \mu_2)^i(A) = \int F_i^A \, d\mu_i$ is well-defined in $\overline{\mathbb{R}}_+$, and from Lemma 824 (*measure of section*), it is equal to $\int \mu_j(s_i(x_i, A)) \, d\mu_i$.

From Lemma 550 (*compatibility of section with set operations*, $\varnothing$), Definition 611 (*measure*, $\mu_j(\varnothing) = 0$), and Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*), we have

$$(\mu_1 \otimes \mu_2)^i(\varnothing) = \int \mu_j(s_i(x_i, \varnothing)) \, d\mu_i = \int \mu_j(\varnothing) \, d\mu_i = \int 0 \, d\mu_i = 0.$$

Let $(A_n)_{n\in\mathbb{N}} \in \Sigma$. Assume that the A_n's are pairwise disjoint. Let $A \stackrel{\text{def.}}{=} \biguplus_{n\in\mathbb{N}} A_n$. Let x_i be in X_i. Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$. Assume that $p \neq q$, Then, from Lemma 551 (*measurability of section*), Lemma 552 (*countable union of sections is measurable*, with $I \stackrel{\text{def.}}{=} \mathbb{N}$), and Lemma 550 (*compatibility of section with set operations*, with intersection and $\varnothing$), $s_i(x_i, A_n) \in \Sigma_j$, and

$$s_i(x_i, A) = s_i\left(x_i, \bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} s_i(x_i, A_n) \in \Sigma_j,$$
$$s_i(x_i, A_p) \cap s_i(x_i, A_q) = s_i(x_i, A_p \cap A_q) = s_i(x_i, \varnothing) = \varnothing.$$

Thus, from Lemma 824 (*measure of section*), Definition 611 (*measure*, μ_j is σ-additive), Definition 608 (*σ-additivity over measurable space*), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j\left(\biguplus_{n\in\mathbb{N}} s_i(x_i, A_n)\right) = \sum_{n\in\mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sum_{n\in\mathbb{N}} F_i^{A_n}(x_i).$$

Then, from Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*), and Definition 830 (*candidate tensor product measure*), we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \sum_{n\in\mathbb{N}} F_i^{A_n} \, d\mu_i = \sum_{n\in\mathbb{N}} \int F_i^{A_n} \, d\mu_i = \sum_{n\in\mathbb{N}} (\mu_1 \otimes \mu_2)^i(A_n).$$

Hence, from Definition 608 (*σ-additivity over measurable space*), $(\mu_1 \otimes \mu_2)^i$ is σ-additive.

Therefore, from Definition 611 (*measure*), $(\mu_1 \otimes \mu_2)^i$ is a measure on (X, Σ).

Identity. Let $A_1 \in \Sigma_1$. Let $A_2 \in \Sigma_2$. Let $A \stackrel{\text{def.}}{=} A_1 \times A_2$.

Then, from Lemma 542 (*product of measurable subsets is measurable*), we have $A \in \Sigma$. Hence, from Definition 830 (*candidate tensor product measure*), Lemma 825 (*measure of section of product*), Definition 611 (*measure*, μ_j is nonnegative), Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*, with $a \stackrel{\text{def.}}{=} \mu_j(A_j) \in \overline{\mathbb{R}}_+$), Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, with $\mu \stackrel{\text{def.}}{=} \mu_i$ and $A_i \in \Sigma_i$), Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*), and since $\{1, 2\} = \{i, j\}$, we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \mu_j(A_j) \, \mathbb{1}_{A_i} \, d\mu_i = \mu_j(A_j) \, \mu_i(A_i) = \mu_1(A_1) \, \mu_2(A_2),$$

Therefore, from Definition 829 (*tensor product measure*), $(\mu_1 \otimes \mu_2)^i$ is a tensor product measure. □

Lemma 832 ***(tensor product of finite measures).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be finite measure spaces. Let μ be a tensor product measure on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ relying on μ_1 and μ_2. Then, μ is finite.

Proof. Direct consequence of Definition 611 (*measure*), Lemma 475 (*equivalent definition of σ-algebra*, $X_1 \in \Sigma_1$, $X_2 \in \Sigma_2$, and $X_1 \times X_2 \in \Sigma_1 \otimes \Sigma_2$), Definition 829 (*tensor product measure*), Definition 622 (*finite measure*), and **closedness of multiplication in $\mathbb{R}_+$**. □

Lemma 833 ***(tensor product of σ-finite measures).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. Let μ be a tensor product measure on (X, Σ) relying on μ_1 and μ_2. Then, μ is σ-finite.
In particular, let $i \in \{1, 2\}$. Let $(B_{i,n})_{n \in \mathbb{N}} \in \Sigma_i$ such that

$$(13.82) \qquad (\forall n \in \mathbb{N}, \quad B_{i,n} \subset B_{i,n+1} \quad \wedge \quad \mu_i(B_{i,n}) < \infty) \quad \wedge \quad X_i = \bigcup_{n \in \mathbb{N}} B_{i,n}.$$

For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$. Then, we have

$$(13.83) \qquad (\forall n \in \mathbb{N}, \quad B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) < \infty) \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} B_n.$$

Proof. Existence of the $B_{i,n}$'s comes from Lemma 625 (*equivalent definition of σ-finite measure*).
Let $n \in \mathbb{N}$. Then, from the definition of the B_n's, Lemma 542 (*product of measurable subsets is measurable*), **monotonicity of Cartesian product**, Definition 829 (*tensor product measure*), and **closedness of multiplication in $\overline{\mathbb{R}}_+$**, we have

$$B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) = \mu_1(B_{1,n})\, \mu_2(B_{2,n}) < \infty.$$

Moreover, from **monotonicity of union**, we have $\bigcup_{n \in \mathbb{N}} B_n \subset X$. Conversely, let (x_1, x_2) be in X. Let $i \in \{1, 2\}$. Then, from Lemma 625 (*equivalent definition of σ-finite measure*, with μ_i), there exists $n_i \in \mathbb{N}$ such that $x_i \in B_{i,n_i}$. Thus, from monotonicity of the $B_{i,n}$'s, we have $x_i \in B_{i,\max(n_1,n_2)}$. Then, from the definition of the B_n's, we have $(x_1, x_2) \in B_{\max(n_1,n_2)}$, i.e. $X \subset \bigcup_{n \in \mathbb{N}} B_n$. Hence, we have $X = \bigcup_{n \in \mathbb{N}} B_n$.
Therefore, from Definition 624 (*σ-finite measure*), μ is σ-finite. □

Remark 834. The next proof follows the monotone class theorem scheme (see Section 4.3).

Lemma 835 ***(uniqueness of tensor product measure (finite)).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be finite measure spaces. Then, there exists a unique tensor product measure on the product measurable space $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$.

Proof. **Existence.** Direct consequence of Lemma 627 (*finite measure is σ-finite*, thus μ_1 and μ_2 are σ-finite), and Lemma 831 (*candidate tensor product measure is tensor product measure*).

Uniqueness. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$, $\overline{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. Let m and $\tilde{m}$ be tensor product measures on (X, Σ) relying on μ_1 and μ_2. Let $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \,|\, m(A) = \tilde{m}(A)\}$. Let $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\overline{\Sigma})$.

(1). $\overline{\Sigma} \subset \mathcal{S}$. Direct consequence of Lemma 542 (*product of measurable subsets is measurable*), Definition 829 (*tensor product measure*, with m and $\tilde{m}$), and the definition of $\mathcal{S}$.

(2). $\mathcal{A} \subset \mathcal{S}$. Let $A \in \mathcal{A}$.
Then, from Definition 541 (*tensor product of σ-algebras*, $\Sigma = \Sigma_X(\overline{\Sigma})$), Lemma 490 (*σ-algebra contains set algebra*, with $G \stackrel{\text{def.}}{=} \overline{\Sigma}$, thus $\mathcal{A} \subset \Sigma$), Lemma 505 (*set algebra generated by product of σ-algebras*), and Lemma 542 (*product of measurable subsets is measurable*, $\overline{\Sigma} \subset \Sigma$), we have $A \in \Sigma$, and there exists $n \in \mathbb{N}$, and $(A_p)_{p \in [0..n]} \in \overline{\Sigma} \subset \Sigma$ such that, for all $p, q \in [0..n]$, $p \neq q$ implies emptiness of $A_p \cap A_q$, and $A = \biguplus_{p \in [0..n]} A_p$. Thus, from Lemma 621 (*equivalent definition of measure*, additivity for m and $\tilde{m}$), Definition 607 (*additivity over measurable space*), (1) ($A_p \in \mathcal{S}$), and the definition of $\mathcal{S}$, and since **addition in $\overline{\mathbb{R}}_+$ is a function (same arguments yield the same result)**, we have

$$m(A) = m\left(\biguplus_{p \in [0..n]} A_p\right) = \sum_{p \in [0..n]} m(A_p) = \sum_{p \in [0..n]} \tilde{m}(A_p) = \tilde{m}\left(\biguplus_{p \in [0..n]} A_p\right) = \tilde{m}(A).$$

Then, from the definition of $\mathcal{S}$, we have $A \in \mathcal{S}$. Hence, we have $\mathcal{A} \subset \mathcal{S}$.

(3). $\mathcal{S}$ is monotone class. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{S} \subset \Sigma$.

Then, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable union and intersection), we have $\bigcup_{n\in\mathbb{N}} A_n, \bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

Assume first that, for all $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Let $A \stackrel{\text{def.}}{=} \bigcup_{n\in\mathbb{N}} A_n$.
Then, from Lemma 617 (*measure is continuous from below*, with m and $\tilde{m}$), the definition of $\mathcal{S}$, and since **supremum is a function**, we have

$$m(A) = m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\in\mathbb{N}} m(A_n) = \sup_{n\in\mathbb{N}} \tilde{m}(A_n) = \tilde{m}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \tilde{m}(A).$$

Hence, from the definition of $\mathcal{S}_i$, we have $A \in \mathcal{S}$, i.e. $\mathcal{S}$ is closed under nondecreasing union.

Assume now that, for all $n \in \mathbb{N}$, $A_n \supset A_{n+1}$. Let $A \stackrel{\text{def.}}{=} \bigcap_{n\in\mathbb{N}} A_n$.
Then, from Lemma 832 (*tensor product of finite measures*, with m and $\tilde{m}$, both relying on finite measures μ_1 and μ_2), Lemma 623 (*finite measure is bounded*, $m(A_0), \tilde{m}(A_0) < \infty$), Lemma 619 (*measure is continuous from above*, with $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$ and $n_0 \stackrel{\text{def.}}{=} 0$), the definition of $\mathcal{S}$, and since **infimum is a function**, we have

$$m(A) = m\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \inf_{n\in\mathbb{N}} m(A_n) = \inf_{n\in\mathbb{N}} \tilde{m}(A_n) = \tilde{m}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \tilde{m}(A).$$

Thus, from the definition of $\mathcal{S}_i$, we have $A \in \mathcal{S}$, i.e. $\mathcal{S}$ is closed under nonincreasing intersection.

Hence, from Definition 448 (*monotone class*), $\mathcal{S}$ is a monotone class.

Therefore, from Definition 541 (*tensor product of σ-algebras*, $\Sigma = \Sigma_X(\overline{\Sigma})$), (2), (3), and Lemma 515 (*usage of monotone class theorem*, with $G \stackrel{\text{def.}}{=} \overline{\Sigma}$), we have $m = \tilde{m}$. □

Remark 836.
The uniqueness part of the following proof reproduces the schemes of the proof of Lemma 828.

Lemma 837 ***(uniqueness of tensor product measure).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$ and $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. Then, there exists a unique tensor product measure on (X, Σ).

This measure is denoted $\mu_1 \otimes \mu_2$, and for all $i \in \{1, 2\}$ with $j \stackrel{\text{def.}}{=} 3 - i$, we have

$$\forall A \in \Sigma, \quad (\mu_1 \otimes \mu_2)(A) = \int \mu_j(s_i(x_i, A))\, d\mu_i. \tag{13.84}$$

It is called the (tensor) product measure *of μ_1 and μ_2, and $(X, \Sigma, \mu_1 \otimes \mu_2)$ is called* (tensor) product measure space.

Proof. **Existence.** Direct consequence of Lemma 831 (*candidate tensor product measure is tensor product measure*).

Uniqueness. Let m and $\tilde{m}$ be tensor product measures on (X, Σ) relying on μ_1 and μ_2.

From Lemma 625 (*equivalent definition of σ-finite measure*), for all $i \in \{1, 2\}$, there exists $(B_{i,n})_{n\in\mathbb{N}} \in \Sigma_i$ such that, for all $n \in \mathbb{N}$, $B_{i,n} \subset B_{i,n+1}$, $\mu_i(B_{i,n}) < \infty$, and $X_i = \bigcup_{n\in\mathbb{N}} B_{i,n}$. For all $n \in \mathbb{N}$, let $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$. Then, from Lemma 833 (*tensor product of σ-finite measures*, with $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$), we have for all $n \in \mathbb{N}$, $B_n \in \Sigma$, $B_n \subset B_{n+1}$, $m(B_n), \tilde{m}(B_n) < \infty$, and $X = \bigcup_{n\in\mathbb{N}} B_n$.

For all $i \in \{1, 2\}$, for all $n \in \mathbb{N}$, from Lemma 475 (*equivalent definition of σ-algebra*, ($A \in \Sigma$ implies $A \cap B_n \in \Sigma$)), and Lemma 629 (*restricted measure*, with $Y \stackrel{\text{def.}}{=} B_{i,n}$ and $\mu \stackrel{\text{def.}}{=} \mu_i$, then

$Y \stackrel{\text{def.}}{=} B_n$ and $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$), let $\mu_{i,n}$, m_n and $\tilde{m}_n$ be the restricted measures defined by

$$\mu_{i,n} \stackrel{\text{def.}}{=} (\mu_i)'_{B_{i,n}} = (A_i \in \Sigma_i \longmapsto \mu_i(A_i \cap B_{i,n})),$$
$$m_n \stackrel{\text{def.}}{=} m'_{B_n} = (A \in \Sigma \longmapsto m(A \cap B_n)) \quad \text{and} \quad \tilde{m}_n \stackrel{\text{def.}}{=} (\tilde{m})'_{B_n} = (A \in \Sigma \longmapsto \tilde{m}(A \cap B_n)).$$

Let $i \in \{1,2\}$. Let $n \in \mathbb{N}$. Then, from Definition 622 (*finite measure*, with $\mu_{i,n}(X_i) = \mu_i(B_{i,n})$, $m_n(X) = m(B_n)$ and $\tilde{m}_n(X) = \tilde{m}(B_n)$ finite), $\mu_{i,n}$, m_n and $\tilde{m}_n$ are finite measures.

Let $n \in \mathbb{N}$. Let $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. Then, from Lemma 542 (*product of measurable subsets is measurable*, $A_1 \times A_2 \in \Sigma$), the definitions of m_n, B_n, $\mu_{1,n}$, $\mu_{2,n}$ and $\tilde{m}_n$, **compatibility of intersection with Cartesian product**, and Definition 829 (*tensor product measure*, with $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$), we have

$$\begin{aligned} m_n(A_1 \times A_2) &= m((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = m((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) \\ &= \mu_1(A_1 \cap B_{1,n})\, \mu_2(A_2 \cap B_{2,n}) = \mu_{1,n}(A_1)\, \mu_{2,n}(A_2) \\ &= \tilde{m}((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) = \tilde{m}((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = \tilde{m}_n(A_1 \times A_2). \end{aligned}$$

Thus, from Definition 829 (*tensor product measure*, with $\mu_1 \stackrel{\text{def.}}{=} \mu_{1,n}$, $\mu_2 \stackrel{\text{def.}}{=} \mu_{2,n}$ and $\mu \stackrel{\text{def.}}{=} m_n, \tilde{m}_n$), m_n and $\tilde{m}_n$ are tensor product measures on (X, Σ) both relying on finite measures $\mu_{1,n}$ and $\mu_{2,n}$. Hence, from Lemma 835 (*uniqueness of tensor product measure (finite)*), we have $m_n = \tilde{m}_n$.

Let $A \in \Sigma$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, $A \cap B_n \in \Sigma$, then $\bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \Sigma$), **distributivity of intersection over union**, **monotonicity of intersection ($(A \cap B_n)_{n \in \mathbb{N}}$ is nondecreasing)**, Lemma 617 (*measure is continuous from below*, with m, $\tilde{m}$ and $A_n \stackrel{\text{def.}}{=} A \cap B_n$), the definition of m_n and $\tilde{m}_n$, and since **supremum is a function**, we have

$$\begin{aligned} m(A) &= m\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = m\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) \\ &= \sup_{n \in \mathbb{N}} m(A \cap B_n) = \sup_{n \in \mathbb{N}} m_n(A) = \sup_{n \in \mathbb{N}} \tilde{m}_n(A) = \sup_{n \in \mathbb{N}} \tilde{m}(A \cap B_n) \\ &= \tilde{m}\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) = \tilde{m}\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = \tilde{m}(A). \end{aligned}$$

Therefore, we have $m = \tilde{m}$.

Identity. Direct consequence of the uniqueness result above, Lemma 831 (*candidate tensor product measure is tensor product measure*), Definition 830 (*candidate tensor product measure*), and Lemma 824 (*measure of section*). □

Lemma 838 ***(negligibility of measurable section).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$. Let $A \in \Sigma_1 \otimes \Sigma_2$. Let $i \in \{1,2\}$. Let $j \stackrel{\text{def.}}{=} 3 - i$. Then, A is μ-negligible iff $\mu(A) = 0$ iff for almost all $x_i \in X_i$, $\mu_j(s_i(x_i, A)) = 0$ iff for almost all $x_i \in X_i$, $s_i(x_i, A)$ is μ_j-negligible.

Proof. Direct consequence of Lemma 551 (*measurability of section*, $s_i(x_i, A) \in \Sigma_j$), Lemma 636 (*negligibility of measurable subset*, with A, then $s_i(x_i, A)$), Lemma 837 (*uniqueness of tensor product measure*), Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*, with μ_i and $(x_i \mapsto \mu_j(s_i(x_i, A)))$). □

13.4.2 Lebesgue measure over product space

Lemma 839 ***(Lebesgue measure on $\mathbb{R}^2$).***
There is a unique measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ that generalizes the area of bounded open boxes.

It is denoted $\lambda^{\otimes 2} \stackrel{\text{def.}}{=} \lambda \otimes \lambda$, and it is called the Lebesgue measure on (Borel subsets of) $\mathbb{R}^2$.

Proof. Direct consequence of Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$), Lemma 728 (*Lebesgue measure is* σ*-finite*), and Lemma 837 (*uniqueness of tensor product measure*). □

Lemma 840 ***(Lebesgue measure on*** $\mathbb{R}^2$ ***generalizes area of boxes).***
Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$*. Assume that* $a_1 \leqslant b_1$ *and* $a_2 \leqslant b_2$*. Then, we have*

$$\lambda^{\otimes 2}([\![a_1, b_1]\!] \times [\![a_2, b_2]\!]) = (b_1 - a_1)\,(b_2 - a_2). \tag{13.85}$$

Proof. Direct consequence of Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$), Definition 829 (*tensor product measure*), Lemma 726 (*Lebesgue measure generalizes length of interval*), and **the definition of the area of boxes of** $\mathbb{R}^2$. □

Lemma 841 ***(Lebesgue measure on*** $\mathbb{R}^2$ ***is zero on lines).***
Let $A \subset \mathbb{R}^2$ *be a box (i.e. the Cartesian product of two intervals).*
Then, we have $\lambda^{\otimes 2}(A) = 0$ *iff* A *is a line (i.e. at least one of the two intervals is a singleton).*

Proof. Direct consequence of Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*, $\underline{\text{with } a_1 = b_1 \text{ or } a_2 = b_2}$), and Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*). □

Lemma 842 ***(Lebesgue measure on*** $\mathbb{R}^2$ ***is*** σ***-finite).***
The measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$ *is* σ*-finite.*

Proof. Direct consequence of Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$), Lemma 728 (*Lebesgue measure is* σ*-finite*), and Lemma 833 (*tensor product of* σ*-finite measures*). □

Lemma 843 ***(Lebesgue measure on*** $\mathbb{R}^2$ ***is diffuse).***
The measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$ *is diffuse.*

Proof. Direct consequence of Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*, $\underline{\text{with } a_1 = b_1 \text{ and } a_2 = b_2}$), and Definition 626 (*diffuse measure*, $\underline{\{(a_1, a_2)\} = [a_1, a_1] \times [a_2, a_2]}$). □

13.4.3 Integration over product space of nonnegative function

Definition 844 ***(partial function of function from product space).***
Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces. Let $f : X_1 \times X_2 \to \overline{\mathbb{R}}_+$.
Let $i \in \{1, 2\}$. Let $j \overset{\text{def.}}{=} 3 - i$. Let $\psi \overset{\text{def.}}{=} ((x_i, x_j) \mapsto (x_1, x_2))$.
For all $x_i \in X_i$, f_{x_i} denotes the function $(x_j \mapsto f \circ \psi(x_i, x_j))$ from X_j to $\overline{\mathbb{R}}_+$.
Moreover, when $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$, $I_{f,i}$ denotes the function $(x_i \mapsto \int f_{x_i}\, d\mu_j)$ from X_i to $\overline{\mathbb{R}}_+$.

Remark 845. The next proof follows steps 1 to 3 of the Lebesgue scheme (see Section 4.1).
See also the sketch of the proof in Section 5.3.

Theorem 846 ***(Tonelli).*** *Let* (X_1, Σ_1, μ_1) *and* (X_2, Σ_2, μ_2) *be* σ*-finite measure spaces.*
Let $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$*. Let* $i \in \{1, 2\}$*. Let* $j \overset{\text{def.}}{=} 3 - i$*.*
Then, for all $x_i \in X_i$*,* $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$*,* $I_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i)$*, and we have*

$$\int f\, d(\mu_1 \otimes \mu_2) = \int I_{f,i}\, d\mu_i. \tag{13.86}$$

Proof. Let $X \overset{\text{def.}}{=} X_1 \times X_2$, $\Sigma \overset{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$, and $\mu \overset{\text{def.}}{=} \mu_1 \otimes \mu_2$.

(1). For $\boldsymbol{f \in \mathcal{IF}(X, \Sigma)}$**.**

Let $A \overset{\text{def.}}{=} \{f \neq 0\}$. Then, from Lemma 733 (*indicator and support are each other inverse*), we have $A \in \Sigma$ and $f = \mathbb{1}_A$. Let $x_i \in X_i$. Then, from Definition 844 (*partial function of function* *from product space*), Lemma 554 (*indicator of section*), Lemma 551 (*measurability of section*, $s_i(x_i, A) \in \Sigma_j$), and Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, with $A \stackrel{\text{def.}}{=} s_i(x_i, A)$ and μ_j), we have

$$f_{x_i} = \mathbb{1}_{s_i(x_i,A)} \in \mathcal{M}_+(X_j, \Sigma_j) \quad \text{and} \quad I_{f,i} = \int \mathbb{1}_{s_i(x_i,A)}\, d\mu_j = \mu_j(s_i(x_i, A)).$$

Hence, from Lemma 824 (*measure of section*), and Lemma 828 (*measurability of measure of section*), we have $I_{f,i} = F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$. Therefore, from Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*, with $\mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ and μ), and Lemma 837 (*uniqueness of tensor product measure*), we have

$$\int f\, d\mu = \mu(A) = \int F_i^A\, d\mu_i = \int I_{f,i}\, d\mu_i.$$

(2). For $f \in \mathcal{SF}_+(X, \Sigma)$.
From Lemma 767 (*$\mathcal{SF}_+$ simple representation*), let $n \in \mathbb{N}$, $(a_k)_{k\in[0..n]} \in \mathbb{R}_+$ and $(A_k)_{k\in[0..n]} \in \Sigma$ such that $f = \sum_{k\in[0..n]} a_k \mathbb{1}_{A_k}$. Let $x_i \in X_i$. Then, from **left linearity of composition**, (1), Lemma 597 (*$\mathcal{M}_+$ is closed under addition*), Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*), and Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*, with μ_j), we have

$$f_{x_i} = \sum_{k\in[0..n]} a_k\, (\mathbb{1}_{A_k})_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j) \quad \text{and} \quad I_{f,i} = \sum_{k\in[0..n]} a_k\, I_{\mathbb{1}_{A_k},i} \in \mathcal{M}_+(X_i, \Sigma_i).$$

Therefore, from Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*, with μ, then μ_i), and (1), we have

$$\int f\, d\mu = \sum_{k\in[0..n]} a_k \left(\int \mathbb{1}_{A_k}\, d\mu \right) = \sum_{k\in[0..n]} a_k \left(\int I_{\mathbb{1}_{A_k},i}\, d\mu_i \right) = \int I_{f,i}\, d\mu_i.$$

(3). For $f \in \mathcal{M}_+(X, \Sigma)$.
From Lemma 799 (*adapted sequence in $\mathcal{M}_+$*), let $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$ be an adapted sequence for f. Let $x_i \in X_i$. Then, from **compatibility of composition with limit**, (2), Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*), and Lemma 800 (*usage of adapted sequences*, with μ_j), we have $f_{x_i} = \lim_{n\to\infty} (\varphi_n)_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$ and $I_{f,i} = \lim_{n\to\infty} I_{\varphi_n,i} \in \mathcal{M}_+(X_i, \Sigma_i)$. Therefore, from Lemma 800 (*usage of adapted sequences*, with μ, then μ_i), we have

$$\int f\, d\mu = \lim_{n\to\infty} \int \varphi_n\, d\mu = \lim_{n\to\infty} \int I_{\varphi_n,i}\, d\mu_i = \int I_{f,i}\, d\mu_i.$$

□

Lemma 847 *(Tonelli over subset)*. *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ-finite measure spaces. Let $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$. Let $A \in \Sigma$. Let $Y \subset X_1 \times X_2$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}_+$. Assume that $f_{|A} \in \mathcal{M}_+(A, \Sigma \overline{\cap} A)$. Let $i \in \{1, 2\}$ with $j \stackrel{\text{def.}}{=} 3 - i$. Let $\psi \stackrel{\text{def.}}{=} ((x_i, x_j) \mapsto (x_1, x_2))$. For all $x_i \in X_i$, let $A_{x_i} \stackrel{\text{def.}}{=} s_i(x_i, A)$, and $f_{x_i}^A \stackrel{\text{def.}}{=} (x_j \in A_{x_i} \mapsto f \circ \psi(x_i, x_j))$.*
Let $I_{f,i}^A \stackrel{\text{def.}}{=} (x_i \in X_i \mapsto \int_{A_{x_i}} f_{x_i}^A\, d\mu_j)$.
Then, for all $x_i \in X_i$, $f_{x_i}^A \in \mathcal{M}_+(A_{x_i}, \Sigma_j \overline{\cap} A_{x_i})$, $I_{f,i}^A \in \mathcal{M}_+(X_i, \Sigma_i)$, and we have

$$\int_A f\, d(\mu_1 \otimes \mu_2) = \int I_{f,i}^A\, d\mu_i. \tag{13.87}$$

Proof. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$. Let $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$. Let $\hat{f} : X \to \overline{\mathbb{R}}_+$ such that $\hat{f}_{|Y} = f$. Then, from Lemma 813 (*integral in $\mathcal{M}_+$ over subset*), we have $\hat{f}\, \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ and $\int_A f\, d\mu = \int \hat{f}\, \mathbb{1}_A\, d\mu$.

Hence, from Theorem 846 (*Tonelli*, with $f \stackrel{\text{def.}}{=} \hat{f}\,\mathbb{1}_A$), and Definition 844 (*partial function of function from product space*), for all $x_i \in X_i$, $(\hat{f}\mathbb{1}_A)_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$, $I_{\hat{f}\mathbb{1}_A,i} \in \mathcal{M}_+(X_i, \Sigma_i)$, and we have $\int \hat{f}\,\mathbb{1}_A\,d\mu = \int I_{\hat{f}\mathbb{1}_A,i}\,d\mu_i$.

Let $x_i \in X_i$. Then, by construction, and from Lemma 554 (*indicator of section*), we have $f^A_{x_i} = (\hat{f}\mathbb{1}_A)_{x_i} = \hat{f}_{x_i}\mathbb{1}_{A_{x_i}}$. Therefore, from Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, with μ_j), we have $f^A_{x_i} \in \mathcal{M}_+(A_{x_i}, \Sigma_j \overline{\cap} A_{x_i})$, $I^A_{f,i} = \int (\hat{f}\mathbb{1}_A)_{x_i}\,d\mu_j = I_{\hat{f}\mathbb{1}_A,i}$, i.e. $I^A_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i)$ and $\int_A f\,d\mu = \int I^A_{f,i}\,d\mu_i$. □

Lemma 848 ***(Tonelli for tensor product).***
For all $i \in \{1,2\}$, let (X_i, Σ_i, μ_i) be a σ-finite measure space, and $f_i \in \mathcal{M}_+(X_i, \Sigma_i)$.
Then, $f_1 \otimes f_2 \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$, and we have

$$\int (f_1 \otimes f_2)\,d(\mu_1 \otimes \mu_2) = \left(\int f_1\,d\mu_1\right)\left(\int f_2\,d\mu_2\right). \tag{13.88}$$

Proof. Let $X \stackrel{\text{def.}}{=} X_1 \times X_2$, $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$, $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$, and $f \stackrel{\text{def.}}{=} f_1 \otimes f_2$.
Then, from Definition 604 (*tensor product of numeric functions*), Lemma 605 (*measurability of tensor product of numeric functions*), Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*), and Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), f belongs to $\mathcal{M}_+(X, \Sigma)$.

Then, from Theorem 846 (*Tonelli*, with $i \stackrel{\text{def.}}{=} 1$), Definition 844 (*partial function of function from product space*), Definition 604 (*tensor product of numeric functions*), Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*, with $f \stackrel{\text{def.}}{=} f_2$ and $a \stackrel{\text{def.}}{=} f_1(x_1) \in \overline{\mathbb{R}}_+$), and Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*), we have

$$\int f\,d\mu = \int I_{f,1}\,d\mu_1 \quad \text{where } \forall x_1 \in X_1,\ I_{f,1}(x_1) \stackrel{\text{def.}}{=} \int f_1(x_1)\,f_2\,d\mu_2 = \left(\int f_2\,d\mu_2\right) f_1(x_1).$$

Therefore, from Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*, with $a \stackrel{\text{def.}}{=} \int f_2\,d\mu_2 \in \overline{\mathbb{R}}_+$ and $f \stackrel{\text{def.}}{=} f_1$), and Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*), we have

$$\int f\,d\mu = \int \left(\int f_2\,d\mu_2\right) f_1\,d\mu_1 = \left(\int f_1\,d\mu_1\right)\left(\int f_2\,d\mu_2\right).$$

□

Chapter 14

Integration of real functions

Contents

Remark 849. From now on, the expressions involving integrals are taken in $\mathbb{R}$, thus functions also take almost all their values in $\mathbb{R}$.

14.1 Definition of the integral

Remark 850. This section follows step 4 of the Lebesgue scheme (see Section 4.1).

Definition 851 ***(integrability).*** Let (X, Σ, μ) be a measure space. A function $f : X \to \overline{\mathbb{R}}$ is said *μ-integrable (in $\mathcal{M}$)* iff f^+ and f^- are μ-integrable in $\mathcal{M}_+$.

Lemma 852 ***(integrable is measurable).*** *Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}$. Assume that f is μ-integrable in $\mathcal{M}$. Then, we have $f \in \mathcal{M}$.*

Proof. Direct consequence of Definition 851 (*integrability*), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of μ-integrability with f^+ and f^-), and Lemma 594 (*measurability of nonnegative and nonpositive parts*). □

Lemma 853 ***(equivalent definition of integrability).*** *Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}$. Then, f is μ-integrable in $\mathcal{M}$ iff $f \in \mathcal{M}$ and $|f|$ is μ-integrable in $\mathcal{M}_+$.*

Proof. **"Left" implies "right".** Direct consequence of Definition 851 (*integrability*), Lemma 852 (*integrable is measurable*), Lemma 596 (*$\mathcal{M}$ is closed under absolute value*), Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*), Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*), and Lemma 789 (*integral in $\mathcal{M}_+$*, $|f|$ is μ-integrable in $\mathcal{M}_+$).

"Right" implies "left". Direct consequence of Lemma 594 (*measurability of nonnegative and nonpositive parts*), Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*), Lemma 321 (*infinity-sum property in $\overline{\mathbb{R}}_+$*, contrapositive), Lemma 789 (*integral in $\mathcal{M}_+$*, f^+ and f^- are μ-integrable in $\mathcal{M}_+$), and Definition 851 (*integrability*).

Therefore, we have the equivalence. □

Lemma 854 ***(compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$).***
Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}$. Assume that f is nonnegative. Then, f is μ-integrable in $\mathcal{M}$ iff f is μ-integrable in $\mathcal{M}_+$.

Proof. Direct consequence of Lemma 853 (*equivalent definition of integrability*), and Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*, $|f| = f$). □

Lemma 855 ***(integrable is almost finite).*** *Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}$ be μ-integrable in $\mathcal{M}$. Then, we have $\mu(f^{-1}(\pm\infty)) = 0$, i.e. $|f| \overset{\mu\text{ a.e.}}{<} \infty$.*

Proof. Direct consequence of Lemma 853 (*equivalent definition of integrability*, $|f|$ is integrable in $\mathcal{M}_+$), Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*, $f^{-1}(\pm\infty) = |f|^{-1}(\infty)$), and Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*, with $|f|$). □

Lemma 856 ***(almost bounded by integrable is integrable).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Then, f is μ-integrable in $\mathcal{M}$ iff there exists $g : X \to \overline{\mathbb{R}}_+$ such that g is μ-integrable in $\mathcal{M}_+$ and $|f| \overset{\mu\text{ a.e.}}{\leqslant} g$.

Proof. **"Left" implies "right".** Direct consequence of Lemma 853 (*equivalent definition of integrability*, with $g \overset{\text{def.}}{=} |f|$), Lemma 658 (*almost order is order relation*, reflexivity).

"Right" implies "left". Direct consequence of Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f| \in \mathcal{M}_+$), Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*, with $|f|$ and g), Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, transitivity), Lemma 789 (*integral in $\mathcal{M}_+$*, $|f|$ is μ-integrable in $\mathcal{M}_+$), and Lemma 853 (*equivalent definition of integrability*).

Therefore, we have the equivalence. □

Lemma 857 ***(bounded by integrable is integrable).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Then, f is μ-integrable in $\mathcal{M}$ iff there exists $g : X \to \overline{\mathbb{R}}_+$ μ-integrable in $\mathcal{M}_+$ such that $|f| \leqslant g$.

Proof. **"Left" implies "right".** Direct consequence of Lemma 853 (*equivalent definition of integrability*, with $g \overset{\text{def.}}{=} |f|$), and Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, reflexivity).

"Right" implies "left". Direct consequence of Lemma 643 (*everywhere implies almost everywhere*), and Lemma 856 (*almost bounded by integrable is integrable*).

Therefore, we have the equivalence. □

Definition 858 ***(integral).*** Let (X, Σ, μ) be a measure space. Let f be μ-integrable in $\mathcal{M}$. The *Lebesgue integral of f (for the measure μ)* is still denoted $\int f\, d\mu$; it is defined by

$$\int f\, d\mu \overset{\text{def.}}{=} \int f^+\, d\mu - \int f^-\, d\mu \quad \in \mathbb{R}. \tag{14.1}$$

Lemma 859 ***(compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$).*** *Let (X, Σ, μ) be a measure space. Let f be μ-integrable in $\mathcal{M}$. Assume that f is nonnegative. Then, both Lemma 789 (integral in $\mathcal{M}_+$), and Definition 858 (integral) provide the same value for the integral of f.*

Proof. Direct consequence of Definition 399 (*nonnegative and nonpositive parts*, $f^+ = f$ and $f^- = 0$), Definition 851 (*integrability*), Lemma 789 (*integral in $\mathcal{M}_+$*, f is μ-integrable in $\mathcal{M}_+$), Definition 858 (*integral*), and Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*). □

Lemma 860 ***(integral of zero is zero).***
Let (X, Σ, μ) be a measure space. Then, 0 is μ-integrable in $\mathcal{M}$, and $\int 0\, d\mu = 0$.

Proof. Direct consequence of Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*), and Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*). □

Definition 861 ***(merge integral in $\mathcal{M}$ and $\mathcal{M}_+$).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. The *integral of f (for the measure μ) exists* iff f is nonnegative (and $\int f \, d\mu \in \overline{\mathbb{R}}_+$), or f is μ-integrable (and $\int f \, d\mu \in \mathbb{R}$).

Lemma 862 ***(compatibility of integral with almost equality).***
Let (X, Σ, μ) be a measure space. Let f be μ-integrable in $\mathcal{M}$. Let $g \in \mathcal{M}$. Assume that $g \overset{\mu \text{ a.e.}}{=} f$. Then, g is μ-integrable in $\mathcal{M}$, and we have

$$\int g \, d\mu = \int f \, d\mu. \tag{14.2}$$

Proof. Direct consequence of Lemma 660 (*compatibility of almost equality with operator*, with the binary operator max and the unary operator additive inverse, $g^+ \overset{\mu \text{ a.e.}}{=} f^+$ and $g^- \overset{\mu \text{ a.e.}}{=} f^-$), Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*), Definition 851 (*integrability*), and Definition 858 (*integral*). □

Remark 863. From Lemma 855, integrable functions can only take infinite values on a negligible subset, and from the previous lemma, the integral keeps the same value when the function is modified on a negligible subset. Therefore, we can restrict the study of integrable functions to the sole case of real-valued functions.

14.2 Notations for specific cases

Lemma 864 ***(integral over subset).***
Let (X, Σ, μ) be a measure space. Let $A \in \Sigma$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, $f_{|A}$ is μ_A-integrable in $\mathcal{M}(A, \Sigma \overline{\cap} A)$ iff $\hat{f}\,\mathbb{1}_A$ is μ-integrable in $\mathcal{M}(X, \Sigma)$. If so, the function f is said μ-integrable over A, *and we have*

$$\int f_{|A}\, d\mu_A = \int \hat{f}\,\mathbb{1}_A\, d\mu. \tag{14.3}$$

This integral is still denoted $\int_A f\, d\mu$; it is still called integral of f over A.

Proof. **Equivalence.** Direct consequence of Lemma 406 (*compatibility of nonpositive and nonnegative parts with restriction*), Lemma 405 (*compatibility of nonpositive and nonnegative parts with mask*, $(\hat{f})^\pm$ is an extension of $f^\pm$ to X), Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, $\int f^\pm{}_{|A}\, d\mu_A$ equals $\int (\hat{f})^\pm \mathbb{1}_A\, d\mu$ and is finite), Definition 851 (*integrability*), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of integrability in $\mathcal{M}_+$), Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, with $f^\pm$ and $\hat{f} \stackrel{\text{def.}}{=} (\hat{f})^\pm$), and **the tautology**

$$\left\{\begin{array}{l} P \Leftrightarrow Q \wedge R \\ \hat{P} \Leftrightarrow \hat{Q} \wedge \hat{R} \\ Q \Leftrightarrow \hat{Q} \\ Q \vee \hat{Q} \Rightarrow (R \Leftrightarrow \hat{R}) \end{array}\right. \quad \Longrightarrow \quad (P \Leftrightarrow \hat{P}).$$

Identity. Direct consequence of Definition 858 (*integral*, with $f_{|A}$, then $\hat{f}\,\mathbb{1}_A$), and Lemma 813 (*integral in $\mathcal{M}_+$ over subset*, with $f \stackrel{\text{def.}}{=} f^\pm$ and $\hat{f} \stackrel{\text{def.}}{=} (\hat{f})^\pm$). □

Lemma 865 ***(integral over subset is σ-additive).***
Let (X, Σ, μ) be a measure space. Let $I \subset \mathbb{N}$. Let $A, (A_i)_{i \in I} \in \Sigma$. Assume that $A = \biguplus_{i \in I} A_i$. Let $Y \subset X$ such that $A \subset Y$. Let $f : Y \to \overline{\mathbb{R}}$. Let $\hat{f} : X \to \overline{\mathbb{R}}$. Assume that $\hat{f}_{|Y} = f$. Then, $\hat{f}\,\mathbb{1}_A \in \mathcal{M}$ iff for all $i \in I$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{M}$.

Moreover, if $\sum_{i \in I} |\hat{f}\,\mathbb{1}_{A_i}|$ is μ-integrable, then f is μ-integrable over A, and we have

$$\int_A f\, d\mu = \sum_{i \in I} \int_{A_i} f\, d\mu. \tag{14.4}$$

Proof. **Equivalence.** Direct consequence of Lemma 853 (*equivalent definition of integrability*), Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*, equivalence, same proof for $\hat{f}\,\mathbb{1}_A \in \mathcal{M}$ iff for all $i \in I$, $\hat{f}\,\mathbb{1}_{A_i} \in \mathcal{M}$ since the countable sum is well-defined).

***f* is *μ*-integrable over *A*.** Direct consequence of **the countable triangle inequality for the absolute value in** $\mathbb{R}$, Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), and Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*, $\int |\hat{f}\,\mathbb{1}_A|\, d\mu \leqslant \sum_{i \in I} \int |\hat{f}\,\mathbb{1}_{A_i}|\, d\mu$).

Identity. Direct consequence of Definition 858 (*integral*, with $f \stackrel{\text{def.}}{=} \hat{f}\,\mathbb{1}_A$), Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*, with f^+ and f^-), **associativity of addition in** $\mathbb{R}$ **(both countable sums are finite)**. □

Lemma 866 ***(integral over singleton).*** *Let (X, Σ, μ) be a measure space. Let $a \in X$. Assume that $\{a\} \in \Sigma$. Let $f : X \to \overline{\mathbb{R}}$. Then, f is μ-integrable over $\{a\}$ iff*

$$(f(a) \text{ and } \mu(\{a\}) \text{ are finite}) \quad \vee \quad f(a) = 0 \quad \vee \quad \mu(\{a\}) = 0. \tag{14.5}$$

If so, we have

$$\int_{\{a\}} f \, d\mu = f(a)\mu(\{a\}). \tag{14.6}$$

Proof. From Lemma 864 (*integral over subset*, μ-integrability over subset), Lemma 853 (*equivalent definition of integrability*), Lemma 789 (*integral in $\mathcal{M}_+$*, μ-integrability in $\mathcal{M}_+$), **nonnegativeness of the indicator function**, Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*, with $|f|$, $|f \mathbb{1}_{\{a\}}|$ is equal to $|f(a)| \mathbb{1}_{\{a\}} \in \mathcal{M}_+$), Lemma 345 (*finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)*), **closedness of absolute value in $\mathbb{R}$**, Definition 297 (*absolute value in $\overline{\mathbb{R}}$*, absolute value is closed in $\{\pm\infty\}$), and Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*), we have

$$\begin{aligned}
f \text{ is } \mu\text{-integrable over } \{a\} \quad &\iff \quad f \mathbb{1}_{\{a\}} \text{ is } \mu\text{-integrable in } \mathcal{M} \\
&\iff \quad |f \mathbb{1}_{\{a\}}| \text{ is } \mu\text{-integrable in } \mathcal{M}_+ \\
&\iff \quad |f \mathbb{1}_{\{a\}}| \in \mathcal{M}_+ \wedge \int |f \mathbb{1}_{\{a\}}| \, d\mu < \infty \\
&\iff \quad |f(a)|\mu(\{a\}) < \infty \\
&\iff \quad (f(a) \in \mathbb{R} \quad \wedge \quad \mu(\{a\}) \in \mathbb{R}_+) \\
&\qquad\quad \vee \quad f(a) = 0 \quad \vee \quad \mu(\{a\}) = 0.
\end{aligned}$$

Assume that f is μ-integrable over $\{a\}$. Then, from Lemma 864 (*integral over subset*, $f \mathbb{1}_{\{a\}}$ is μ-integrable in $\mathcal{M}$), Definition 851 (*integrability*), Definition 399 (*nonnegative and nonpositive parts*), and **nonnegativeness of the indicator function**,

$$(f \mathbb{1}_{\{a\}})^+ = f^+ \mathbb{1}_{\{a\}} \text{ and } (f \mathbb{1}_{\{a\}})^- = f^- \mathbb{1}_{\{a\}} \text{ are } \mu\text{-integrable in } \mathcal{M}_+.$$

Moreover, from Lemma 864 (*integral over subset*), Definition 858 (*integral*), Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*), and Lemma 403 (*decomposition into nonnegative and nonpositive parts*, $f(a) = f^+(a) - f^-(a)$), we have

$$\begin{aligned}
\int_{\{a\}} f \, d\mu = \int f \mathbb{1}_{\{a\}} \, d\mu &= \int (f \mathbb{1}_{\{a\}})^+ \, d\mu - \int (f \mathbb{1}_{\{a\}})^- \, d\mu \\
&= f^+(a)\mu(\{a\}) - f^-(a)\mu(\{a\}) = f(a)\mu(\{a\}).
\end{aligned}$$

□

Lemma 867 ***(integral over interval).***
Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a measure space. Assume that μ is diffuse. Let $a, b \in \overline{\mathbb{R}}$ such that $a \leqslant b$. Let f be a μ-integrable function over (a, b). Then, the integral over the interval remains the same when the interval is closed at one or both of its finite extremities.

The integral of f over an interval with extremities a and b is called integral of f from a to b; *it is denoted $\int_a^b f(x) \, d\mu(x)$. When $a < b$, we assume the convention $\int_b^a f(x) \, d\mu(x) \stackrel{\text{def.}}{=} -\int_a^b f(x) \, d\mu(x)$.*

Proof. Direct consequence of Definition 626 (*diffuse measure*), Lemma 865 (*integral over subset is σ-additive*, on $[a, b] = \{a\} \uplus (a, b) \uplus \{b\}$), Lemma 518 (*some Borel subsets*, singletons are measurable), Lemma 866 (*integral over singleton*), and Definition 626 (*diffuse measure*). □

Lemma 868 ***(Chasles relation, integral over split intervals).***
Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ *be a measure space. Assume that* μ *is diffuse. Let* $a, b, c \in \overline{\mathbb{R}}$. *Let* f *be a* μ*-integrable function over* $(\min(a,b,c), \max(a,b,c))$*. Then, we have*

$$\int_a^b f(x)\,d\mu(x) = \int_a^c f(x)\,d\mu(x) + \int_c^b f(x)\,d\mu(x). \tag{14.7}$$

Proof. Direct consequence of Lemma 865 (*integral over subset is σ-additive*). Lemma 867 (*integral over interval*, convention for reverse bounds), and **field properties of** $\mathbb{R}$.

It is trivial when $a \leqslant c \leqslant b$. For instance, when $c < b < a$, we have

$$\int_c^a f(x)\,d\mu(x) = \int_c^b f(x)\,d\mu(x) + \int_b^a f(x)\,d\mu(x).$$

Thus,

$$\begin{aligned}\int_a^b f(x)\,d\mu(x) &= -\int_b^a f(x)\,d\mu(x)\\ &= -\int_c^a f(x)\,d\mu(x) + \int_c^b f(x)\,d\mu(x)\\ &= \int_a^c f(x)\,d\mu(x) + \int_c^b f(x)\,d\mu(x).\end{aligned}$$

□

Lemma 869 ***(integral for counting measure).***
Let (X, Σ) *be a measurable space. Let* $Y \subset X$. *Let* $f \in \mathcal{M}$. *Assume that* $\sum_{y\in Y} |f(y)|$ *is finite. Then,* f *is* δ_Y*-integrable in* $\mathcal{M}$*, and we have*

$$\int f\,d\delta_Y = \sum_{y\in Y} f(y). \tag{14.8}$$

Proof. From Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*, $\int |f|\,d\delta_Y$ is finite), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of μ-integrability in $\mathcal{M}_+$), and Lemma 853 (*equivalent definition of integrability*), f is δ_Y-integrable in $\mathcal{M}$. Then, from Definition 858 (*integral*), Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*), Lemma 403 (*decomposition into nonnegative and nonpositive parts*), and **associativity and commutativity of (possibly uncountable) addition for absolutely convergent sums**, we have

$$\int f\,d\delta_Y = \int f^+\,d\delta_Y - \int f^-\,d\delta_Y = \sum_{y\in Y} f^+(y) - \sum_{y\in Y} f^-(y) = \sum_{y\in Y} f(y).$$

□

Lemma 870 ***(integral for counting measure on*** $\mathbb{N}$***).*** *Let* $f : \mathbb{N} \to \overline{\mathbb{R}}$ *be a sequence. Assume that* $\sum_{n\in\mathbb{N}} |f(n)|$ *is finite. Then,* f *is* δ_Y*-integrable in* $\mathcal{M}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$*, and we have*

$$\int f\,d\delta_{\mathbb{N}} = \sum_{n\in\mathbb{N}} f(n). \tag{14.9}$$

Proof. Direct consequence of Lemma 869 (*integral for counting measure*, with $Y = X \overset{\text{def.}}{=} \mathbb{N}$ and $\Sigma \overset{\text{def.}}{=} \mathcal{P}(\mathbb{N})$). □

Remark 871. Note that the previous lemma makes absolutely convergent series be Lebesgue integrals for the counting measure on natural numbers. Thus, the theory of absolutely converging series can be derived from the theory of Lebesgue integration.

Lemma 872 ***(integral for Dirac measure).*** *Let (X, Σ) be a measurable space. Let $\{a\} \in \Sigma$. Let $f \in \mathcal{M}$. Assume that $f(a)$ is finite. Then, f is δ_a-integrable in $\mathcal{M}$, and we have*

$$\int f \, d\delta_a = f(a). \tag{14.10}$$

Proof. Direct consequence of Definition 675 (*Dirac measure*), and Lemma 869 (*integral for counting measure*, with $Y \stackrel{\text{def.}}{=} \{a\}$). □

Definition 873 ***(integral for Lebesgue measure on $\mathbb{R}$).***
Let $a, b \in \overline{\mathbb{R}}$. Let Y be an interval with extremities $\min(a, b)$ and $\max(a, b)$. Let $f : Y \to \mathbb{R}$. Assume that f is λ-integrable over Y (for the Lebesgue measure λ). The integral of f from a to b is denoted $\int_a^b f(x)\, dx$.

14.3 The seminormed vector space $\mathcal{L}^1$

Lemma 874 ***(seminorm $\mathcal{L}^1$).*** *Let (X, Σ, μ) be a measure space. The function*

$$N_1 \stackrel{\text{def.}}{=} \left(f \in \mathcal{M} \longmapsto \int |f| \, d\mu \in \overline{\mathbb{R}}_+ \right) \tag{14.11}$$

is well-defined; it is called seminorm $\mathcal{L}^1$.

Proof. Direct consequence of Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f| \in \mathcal{M}_+$), Definition 861 (*merge integral in $\mathcal{M}$ and $\mathcal{M}_+$*), and Lemma 789 (*integral in $\mathcal{M}_+$*, nonnegativeness). □

Remark 875.
The function N_1 is shown below to be a seminormed on the vector space $\mathcal{L}^1$, hence its name.

Lemma 876 ***(integrable is finite seminorm $\mathcal{L}^1$).*** *Let (X, Σ, μ) be a measure space. Let $f \in X \to \overline{\mathbb{R}}$. Then, f is μ-integrable in $\mathcal{M}$ iff $f \in \mathcal{M}$ and $N_1(f) < \infty$.*

Proof. **"Left" implies "right".** Direct consequence of Lemma 853 (*equivalent definition of integrability*), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of μ-integrability), and Lemma 874 (*seminorm $\mathcal{L}^1$*).

"Right" implies "left". Direct consequence of Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f| \in \mathcal{M}_+$), Lemma 874 (*seminorm $\mathcal{L}^1$*), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of μ-integrability), and Lemma 853 (*equivalent definition of integrability*). □

Lemma 877 ***(compatibility of N_1 with almost equality).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f \stackrel{\mu\,\text{a.e.}}{=} g$. Then, we have $N_1(f) = N_1(g)$.*

Proof. Direct consequence of Lemma 874 (*seminorm $\mathcal{L}^1$*), Lemma 660 (*compatibility of almost equality with operator*, with the unary operator absolute value), and Lemma 862 (*compatibility of integral with almost equality*). □

Lemma 878 ***(N_1 is almost definite).***
Let (X, Σ, μ) be a measure space. Then, N_1 is almost definite:

$$\forall f \in \mathcal{M}, \quad N_1(f) = 0 \quad \Longleftrightarrow \quad f \stackrel{\mu\,\text{a.e.}}{=} 0. \tag{14.12}$$

Proof. Direct consequence of Lemma 874 (*seminorm $\mathcal{L}^1$*), Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*), and Lemma 686 (*absolute value is almost definite*). □

Lemma 879 ***(N_1 is absolutely homogeneous).***
Let (X, Σ, μ) be a measure space. Then, N_1 is absolutely homogeneous of degree 1:

$$\forall \lambda \in \overline{\mathbb{R}}, \forall f \in \mathcal{M}, \quad N_1(\lambda f) = |\lambda| N_1(f). \tag{14.13}$$

Proof. Direct consequence of Lemma 874 (*seminorm $\mathcal{L}^1$*), **multiplicativity of the absolute value**, Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*), and Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*). □

Lemma 880 ***(integral is homogeneous).*** *Let (X, Σ, μ) be a measure space. Let f be μ-integrable in $\mathcal{M}$. Let $a \in \mathbb{R}$. Then, af is μ-integrable in $\mathcal{M}$, and we have*

$$\int af \, d\mu = a \int f \, d\mu. \tag{14.14}$$

Proof. From Lemma 852 (*integrable is measurable*, $f \in \mathcal{M}$), Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*, $af \in \mathcal{M}$), Lemma 879 (*N_1 is absolutely homogeneous*, $N_1(af) = |a|N_1(f) < \infty$), Lemma 874 (*seminorm $\mathcal{L}^1$*), Lemma 789 (*integral in $\mathcal{M}_+$*, definition of μ-integrability in $\mathcal{M}_+$), and Lemma 853 (*equivalent definition of integrability*), af is μ-integrable in $\mathcal{M}$. Hence, from Lemma 594 (*measurability of nonnegative and nonpositive parts*), Definition 399 (*nonnegative and nonpositive parts*), and **ordered set properties of $\overline{\mathbb{R}}$**, we have $f^\pm \in \mathcal{M}_+$,

$$(af)^\pm = af^\pm \in \mathcal{M}_+ \text{ when } a \geqslant 0, \quad \text{and} \quad (af)^\pm = -af^\mp \in \mathcal{M}_+ \text{ when } a < 0.$$

Therefore, from Definition 858 (*integral*, with af), Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*, with $af^\pm$ or $-af^\mp$), **field properties of $\mathbb{R}$ (all integrals below are finite)**, and Definition 858 (*integral*, with f), we have in both cases

$$\int af\,d\mu = \int (af)^+\,d\mu - \int (af)^-\,d\mu = a\int f^+\,d\mu - a\int f^-\,d\mu = a\int f\,d\mu.$$

□

Remark 881. In the next two lemmas, the summability domain $\mathcal{D}^+(f,g)$ and the almost sum $f \overset{\mu\,\text{a.e.}}{+} g$ are respectively defined in Definition 678, and Lemma 682.

Lemma 882 ***(Minkowski inequality in $\mathcal{M}$).*** *Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f+g$ is well-defined almost everywhere. Let $f', g' \in \mathcal{M}$. Assume that $f' \overset{\mu\,\text{a.e.}}{=} f$, $g' \overset{\mu\,\text{a.e.}}{=} g$, and $\mathcal{D}^+(f', g') = X$. Then, $f'+g', f \overset{\mu\,\text{a.e.}}{+} g \in \mathcal{M}$, and we have*

$$N_1(f'+g') = N_1(f \overset{\mu\,\text{a.e.}}{+} g) \leqslant N_1(f) + N_1(g). \tag{14.15}$$

Proof. From Lemma 682 (*almost sum*), and Lemma 683 (*compatibility of almost sum with almost equality*), we have $f \overset{\mu\,\text{a.e.}}{+} g, f'+g' \in \mathcal{M}$ and $f'+g' \overset{\mu\,\text{a.e.}}{=} f \overset{\mu\,\text{a.e.}}{+} g$. Then, from Lemma 596 (*$\mathcal{M}$ is closed under absolute value*), we have $|f|, |g|, |f'|, |g'|, |f'+g'| \in \mathcal{M}_+$. Therefore, from Lemma 877 (*compatibility of N_1 with almost equality*), **symmetry of equality**, Lemma 874 (*seminorm $\mathcal{L}^1$*, with $f'+g'$), Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*), Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*), Lemma 801 (*integral in $\mathcal{M}_+$ is additive*), Lemma 874 (*seminorm $\mathcal{L}^1$*, with f' and g'), and Lemma 877 (*compatibility of N_1 with almost equality*), we have

$$\begin{aligned} N_1(f \overset{\mu\,\text{a.e.}}{+} g) &= N_1(f'+g') = \int |f'+g'|\,d\mu \\ &\leqslant \int (|f'|+|g'|)\,d\mu = \int |f'|\,d\mu + \int |g'|\,d\mu = N_1(f') + N_1(g') = N_1(f) + N_1(g). \end{aligned}$$

□

Lemma 883 ***(integral is additive).***
Let (X, Σ, μ) be a measure space. Let f, g be μ-integrable in $\mathcal{M}$. Let $f', g' \in \mathcal{M}$. Assume that $f' \overset{\mu\,\text{a.e.}}{=} f$, $g' \overset{\mu\,\text{a.e.}}{=} g$, and $\mathcal{D}^+(f', g') = X$. Then, $f+g$ is well-defined almost everywhere, $f'+g'$ and $f \overset{\mu\,\text{a.e.}}{+} g$ are μ-integrable in $\mathcal{M}$, and we have

$$\int (f'+g')\,d\mu = \int (f \overset{\mu\,\text{a.e.}}{+} g)\,d\mu = \int f\,d\mu + \int g\,d\mu. \tag{14.16}$$

Proof. From Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*), $f, g \in \mathcal{M}$, and $N_1(f), N_1(g) < \infty$. From Lemma 855 (*integrable is almost finite*), and Definition 282 (*addition in $\overline{\mathbb{R}}$*), $f+g$ is well-defined almost everywhere. Then, from Lemma 882 (*Minkowski inequality in $\mathcal{M}$*), and **closedness of addition in $\mathbb{R}_+$**, we have $f'+g', f \overset{\mu\,\text{a.e.}}{+} g \in \mathcal{M}$, and $N_1(f'+g') = N_1(f \overset{\mu\,\text{a.e.}}{+} g) < \infty$. Hence, from Lemma 876 (*integrable is finite seminorm* $\mathcal{L}^1$), $f' + g'$ and $f \overset{\mu\text{ a.e.}}{+} g$ are μ-integrable in $\mathcal{M}$. Moreover, from Lemma 862 (*compatibility of integral with almost equality*), f' and g' are also μ-integrable in $\mathcal{M}$.

From Lemma 594 (*measurability of nonnegative and nonpositive parts*), we have

$$f'^{\pm}, g'^{\pm}, (f' + g')^{\pm} \in \mathcal{M}_+.$$

Therefore, from Lemma 862 (*compatibility of integral with almost equality*), **symmetry of equality**, Definition 858 (*integral*, with $f' + g'$), Lemma 805 (*compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts*, with f' and g'), **field properties of $\mathbb{R}$ (all integrals below are finite)**, Definition 858 (*integral*, with f' and g'), and Lemma 862 (*compatibility of integral with almost equality*, with f and g), we have

$$\begin{aligned}
\int (f \overset{\mu\text{ a.e.}}{+} g)\, d\mu = \int (f' + g')\, d\mu &= \int (f' + g')^+\, d\mu - \int (f' + g')^-\, d\mu \\
&= \int f'^+\, d\mu + \int g'^+\, d\mu - \left(\int f'^-\, d\mu + \int g'^-\, d\mu \right) \\
&= \left(\int f'^+\, d\mu - \int f'^-\, d\mu \right) + \left(\int g'^+\, d\mu - \int g'^-\, d\mu \right) \\
&= \int f'\, d\mu + \int g'\, d\mu = \int f\, d\mu + \int g\, d\mu.
\end{aligned}$$

□

Definition 884 ***($\mathcal{L}^1$, vector space of integrable functions).***
Let (X, Σ, μ) be a measure space. The *vector space of integrable functions* is denoted $\mathcal{L}^1(X, \Sigma, \mu)$ (or simply $\mathcal{L}^1$); it is defined by

(14.17) $$\mathcal{L}^1(X, \Sigma, \mu) \overset{\text{def.}}{=} \{ f \in \mathcal{M}_{\mathbb{R}} \mid N_1(f) < \infty \}.$$

Remark 885. The set $\mathcal{L}^1$ is shown below to be a (seminormed) vector space, hence its name.

Lemma 886 ***(equivalent definition of $\mathcal{L}^1$).*** *Let (X, Σ, μ) be a measure space. Let $f : X \to \overline{\mathbb{R}}$. Then, $f \in \mathcal{L}^1$ iff f is finite and μ-integrable in $\mathcal{M}$.*

Proof. **"Left" implies "right".** Direct consequence of Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $f \in \mathcal{M} \cap \mathbb{R}^X$), Lemma 874 (*seminorm $\mathcal{L}^1$*), Lemma 789 (*integral in $\mathcal{M}_+$*, $|f|$ is μ-integrable in $\mathcal{M}_+$), and Lemma 853 (*equivalent definition of integrability*).

"Right" implies "left". Direct consequence of Lemma 853 (*equivalent definition of integrability*), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $f \in \mathcal{M}_{\mathbb{R}}$), Lemma 789 (*integral in $\mathcal{M}_+$*), Lemma 874 (*seminorm $\mathcal{L}^1$*), and Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*).

Therefore, we have the equivalence. □

Lemma 887 ***(Minkowski inequality in $\mathcal{L}^1$).***
Let (X, Σ, μ) be a measure space. Let $f, g \in \mathcal{L}^1$. Then, we have $N_1(f + g) \leqslant N_1(f) + N_1(g)$.

Proof. Direct consequence of Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*, f and g belong to $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$), Definition 678 (*summability domain*, $\mathcal{D}^+(f, g) = X$), Lemma 685 (*almost sum is sum*), Lemma 643 (*everywhere implies almost everywhere*), and Lemma 882 (*Minkowski inequality in $\mathcal{M}$*). □

Lemma 888 ***($\mathcal{L}^1$ is seminormed vector space).***
Let (X, Σ, μ) be a measure space. Then, $(\mathcal{L}^1, N_1)$ is a seminormed vector space.

Proof. From Lemma 574 (*$\mathcal{M}_{\mathbb{R}}$ is vector space*, with $0 : X \to \mathbb{R}$ as zero), Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), Lemma 643 (*everywhere implies almost everywhere*), and Lemma 878 (*N_1 is almost definite*, $N_1(0) = 0$), $\mathcal{L}^1$ is a subset of the vector space $\mathcal{M}_{\mathbb{R}}$ containing the zero, and we have $N_1(\mathcal{L}^1) \subset \mathbb{R}$.

Let $a \in \mathbb{R}$. Let $f, g \in \mathcal{L}^1 \subset \mathcal{M}_{\mathbb{R}}$. Then, from Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), we have $N_1(f), N_1(g) < \infty$. Moreover, from Lemma 574 (*$\mathcal{M}_{\mathbb{R}}$ is vector space*), we have $af, f+g \in \mathcal{M}_{\mathbb{R}}$. Thus, from Lemma 879 (*N_1 is absolutely homogeneous*), Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*), and **closedness of multiplication and addition in** $\mathbb{R}$, we have

$$N_1(af) = |a| N_1(f) < \infty \quad \text{and} \quad N_1(f+g) \leqslant N_1(f) + N_1(g) < \infty$$

Hence, from Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), we have $af, f+g \in \mathcal{L}^1$, N_1 is absolutely homogeneous of degree 1, and N_1 satisfies the triangle inequality.

Therefore, from Lemma 81 (*closed under vector operations is subspace*, $\mathcal{L}^1$ is a vector subspace of $\mathcal{M}_{\mathbb{R}}$), Definition 77 (*subspace*, $\mathcal{L}^1$ is a vector space), and Definition 237 (*seminorm*, N_1 is a seminorm over $\mathcal{L}^1$), $(\mathcal{L}^1, N_1)$ is a seminormed vector space. □

Definition 889 ***(convergence in $\mathcal{L}^1$).*** Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}}, f \in \mathcal{L}^1$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said *convergent towards f in $\mathcal{L}^1$* iff $\lim_{n\to\infty} N_1(f_n - f) = 0$.

Lemma 890 ***($\mathcal{L}^1$ is closed under absolute value).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{L}^1$. Then, we have $|f| \in \mathcal{L}^1$.

Proof. Direct consequence of Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), Lemma 596 (*$\mathcal{M}$ is closed under absolute value*, $|f| \in \mathcal{M}_{\mathbb{R}}$), **idempotent law for the absolute value**, and Lemma 874 (*seminorm $\mathcal{L}^1$*, $N_1(|f|) = N_1(f)$). □

Lemma 891 ***(bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$).***
Let (X, Σ, μ) be a measure space. Let $f \in \mathcal{M}$. Then, we have the equivalence

$$f \in \mathcal{L}^1 \iff \exists g \in \mathcal{L}^1, \quad |f| \leqslant g. \tag{14.18}$$

Proof. **"Left" implies "right".** Direct consequence of Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*, $g \stackrel{\text{def.}}{=} |f| \in \mathcal{L}^1$), and **reflexivity of order in** $\mathbb{R}$.

"Right" implies "left". From Lemma 886 (*equivalent definition of $\mathcal{L}^1$*, with g), Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*), Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*, transitivity, thus $|f|$ is finite and g is nonnegative), Lemma 301 (*finite absolute value in $\overline{\mathbb{R}}$*), and Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*, with g), we have f finite and g μ-integrable in $\mathcal{M}_+$. Hence, from Lemma 857 (*bounded by integrable is integrable*, f μ-integrable in $\mathcal{M}$), and Lemma 886 (*equivalent definition of $\mathcal{L}^1$*, with f), we have $f \in \mathcal{L}^1$.

Therefore, we have the equivalence. □

Lemma 892 ***(integral is positive linear form on $\mathcal{L}^1$).***
Let (X, Σ, μ) be a measure space. Let $\mathcal{I} : \mathcal{L}^1 \to \mathbb{R}$ be the function defined by for all $f \in \mathcal{L}^1$, $\mathcal{I}(f) \stackrel{\text{def.}}{=} \int f \, d\mu$. Then, $\mathcal{I}$ is a positive linear form on $\mathcal{L}^1$; hence it is nondecreasing.

Moreover, for all $f \in \mathcal{L}^1$, we have $|\mathcal{I}(f)| \leqslant \mathcal{I}(|f|) = N_1(f)$.

Proof. **(1). Linearity.** Direct consequence of Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*), Lemma 880 (*integral is homogeneous*), Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*, $\mathcal{L}^1 \subset \mathcal{M}_{\mathbb{R}}$), Definition 282 (*addition in $\overline{\mathbb{R}}$*, addition is well-defined in $\mathcal{L}^1$), and Lemma 883 (*integral is additive*, with $h \stackrel{\text{def.}}{=} f + g$).

(2). Nonnegativeness. Let $f \in \mathcal{L}^1$. Assume that $f \geqslant 0$. Then, from Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*, $f \in \mathcal{M}_{\mathbb{R}}$), Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*, $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$), and

Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*), we have $f \in \mathcal{M}_+$. Hence, from the definition of $\mathcal{I}$, and Lemma 789 (*integral in $\mathcal{M}_+$*), we have $\mathcal{I}(f) = \int f \, d\mu \geqslant 0$.

Therefore, from Definition 64 (*linear map*), and Definition 66 (*linear form*), $\mathcal{I}$ is a nonnegative linear form on $\mathcal{L}^1$.

(3). Monotonicity. Let $f, g \in \mathcal{L}^1$. Assume that $f \leqslant g$. Then, from Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*), Definition 61 (*vector space*, additive abelian group properties), and **ordered field properties of $\mathbb{R}$**, we have $g = f + (g - f)$ with $g - f \geqslant 0$. Thus, from the definition of $\mathcal{I}$ (with f), (2), (1), and the definition of $\mathcal{I}$ (with g), we have

$$\mathcal{I}(f) = \int f \, d\mu \leqslant \int f \, d\mu + \int (g - f) \, d\mu = \int g \, d\mu = \mathcal{I}(g).$$

Hence, $\mathcal{I}$ is nondecreasing.

(4). Inequality. Let $f \in \mathcal{L}^1$. Then, from Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), Lemma 874 (*seminorm $\mathcal{L}^1$*), Definition 851 (*integrability*), and Lemma 853 (*equivalent definition of integrability*), f^+, f^- and $|f|$ are μ-integrable in $\mathcal{M}_+$, i.e. all integrals below are finite. Hence, from the definition of $\mathcal{I}$ (with f), Definition 858 (*integral*), **the triangle inequality for the absolute value in $\mathbb{R}$**, Lemma 401 (*nonnegative and nonpositive parts are nonnegative*), (2), Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*), the definition of $\mathcal{I}$ (with $|f|$), and Lemma 874 (*seminorm $\mathcal{L}^1$*), we have

$$|\mathcal{I}(f)| = \left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leqslant \int f^+ \, d\mu + \int f^- \, d\mu = \int |f| \, d\mu = \mathcal{I}(|f|) = N_1(f).$$

□

Lemma 893 ***(constant function is $\mathcal{L}^1$).***
Let (X, Σ, μ) be a finite measure space. Let $a \in \mathbb{R}$. Then, $f \stackrel{\text{def.}}{=} (x \mapsto a) \in \mathcal{L}^1$, and we have

$$\int f \, d\mu = a\mu(X). \qquad (14.19)$$

Proof. Direct consequence of Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, $X \in \Sigma$), Definition 748 (*$\mathcal{SF}$, vector space of simple functions*, $|f| = |a| \mathbb{1}_X \in \mathcal{SF}_+$), Lemma 770 (*integral in $\mathcal{SF}_+$*, $\int |f| \, d\mu = |a|\mu(X) < \infty$), Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*), Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*), and Definition 858 (*integral*, $\int f \, d\mu = \operatorname{sgn}(a) \int |f| \, d\mu = a\mu(X)$). □

Lemma 894 ***(first mean value theorem).*** *Let (X, Σ, μ) be a finite measure space. Assume that μ is nonzero. Let $f \in \mathcal{M}$. Assume that f is bounded. Then, $f \in \mathcal{L}^1$, and we have*

$$\inf(f(X)) \leqslant \frac{1}{\mu(X)} \int f \, d\mu \leqslant \sup(f(X)). \qquad (14.20)$$

Moreover, inequalities are strict iff f is not equal to one of its bounds μ-almost everywhere.

Proof. Let $m \stackrel{\text{def.}}{=} \inf(f(X))$ and $M = \sup(f(X))$.

Case $\boldsymbol{0 \leqslant m \leqslant M}$. Then, $-m \leqslant M$ and $|f| \leqslant M$. **Case $\boldsymbol{m < 0 \leqslant M}$.** Then, $|f|$ is less than or equal to $\max(-m, M)$. **Case $\boldsymbol{m \leqslant M < 0}$.** Then, $M \leqslant -m$ and $|f| \leqslant -m$. Thus, in all cases, we have $|f| \leqslant g$ where $g \stackrel{\text{def.}}{=} \max(-m, M)$. Hence, from Lemma 893 (*constant function is $\mathcal{L}^1$*, $g \in \mathcal{L}^1$), and Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*), we have $f \in \mathcal{L}^1$. Therefore, from Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*, $\mathcal{I}$ is nondecreasing), and **ordered field properties of $\mathbb{R}$ with $\boldsymbol{0 < \mu(X) < \infty}$**, we have

$$m \leqslant \frac{1}{\mu(X)} \int f \, d\mu \leqslant M.$$

Moreover, from Lemma 893 (*constant function is* $\mathcal{L}^1$, $(x \mapsto m) \in \mathcal{L}^1$), Lemma 888 ($\mathcal{L}^1$ *is seminormed vector space*, $f - m \in \mathcal{L}^1$), Lemma 892 (*integral is positive linear form on* $\mathcal{L}^1$, $\int (f-m)\,d\mu$ is zero), Lemma 859 (*compatibility of integral in* $\mathcal{M}$ *and* $\mathcal{M}_+$), Lemma 806 (*integral in* $\mathcal{M}_+$ *is almost definite*, with $f - m \geqslant 0$), and Lemma 660 (*compatibility of almost equality with operator*, with the unary operator translation by m), we have

$$\int f\,d\mu = m\mu(X) \quad\Longleftrightarrow\quad \int (f-m)\,d\mu = 0 \quad\Longleftrightarrow\quad f - m \overset{\mu\text{ a.e.}}{=} 0 \quad\Longleftrightarrow\quad f \overset{\mu\text{ a.e.}}{=} m.$$

Similarly, with $M - f \in \mathcal{L}^1 \cap \mathcal{M}_+$, we have $\int f\,d\mu = M\mu(X) \Leftrightarrow f \overset{\mu\text{ a.e.}}{=} M$. Hence, since $(P \Leftrightarrow Q) \Leftrightarrow (\neg P \Leftrightarrow \neg Q)$, we have strict inequalities in (14.20) iff f is not equal to m or M μ-almost everywhere. □

Lemma 895 ***(variant of first mean value theorem).***
Let (X, Σ, μ) *be a finite measure space. Assume that* μ *is nonzero. Let* $f \in \mathcal{M}$, $m \overset{\text{def.}}{=} \inf(f(X))$ *and* $M \overset{\text{def.}}{=} \sup(f(X))$. *Assume that* f *is bounded, not equal to* m *or* M μ*-almost everywhere, and* $(m, M) \subset f(X)$. *Then,* $f \in \mathcal{L}^1$, *and we have*

$$\exists x \in X, \quad \int f\,d\mu = f(x)\mu(X). \tag{14.21}$$

Proof. Direct consequence of Lemma 894 (*first mean value theorem*, with strict inequalities), **ordered field properties of $\mathbb{R}$ with $0 < \mu(X) < \infty$**, and Definition 241 (*interval*, with $X \overset{\text{def.}}{=} \mathbb{R}$, there exists $y \in (m, M)$ such that $\int f\,d\mu = y\mu(X)$). □

Remark 896. See the sketch of next proof in Section 5.2.

Theorem 897 ***(Lebesgue, dominated convergence).***
Let (X, Σ, μ) *be a measure space. Let* $(f_n)_{n\in\mathbb{N}} \in \mathcal{M}$. *Assume that the sequence is pointwise convergent towards* f. *Let* $g \in \mathcal{L}^1$. *Assume that for all* $n \in \mathbb{N}$, $|f_n| \leqslant g$.
Then, for all $n \in \mathbb{N}$, $f_n \in \mathcal{L}^1$, $f \in \mathcal{L}^1$, *the sequence is convergent towards* f *in* $\mathcal{L}^1$, *and we have*

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu. \tag{14.22}$$

Proof. From Lemma 590 ($\mathcal{M}$ *is closed under limit when pointwise convergent*), we have $f \in \mathcal{M}$. Moreover, from Lemma 317 (*absolute value in* $\overline{\mathbb{R}}$ *is continuous*), and **monotonicity of the limit in $\overline{\mathbb{R}}$**, we have $|f| \leqslant g$.

Let $n \in \mathbb{N}$. Then, from Lemma 891 (*bounded by* $\mathcal{L}^1$ *is* $\mathcal{L}^1$), we have $f_n, f \in \mathcal{L}^1$. Thus, from Lemma 888 ($\mathcal{L}^1$ *is seminormed vector space*), Definition 237 (*seminorm*), Definition 61 (*vector space*, $(\mathcal{L}^1, +)$ is an abelian group), Lemma 890 ($\mathcal{L}^1$ *is closed under absolute value*), **linearity and compatibility of the limit with the absolute value**, Lemma 302 (*absolute value in* $\overline{\mathbb{R}}$ *is nonnegative*), and Lemma 304 (*absolute value in* $\overline{\mathbb{R}}$ *is definite*), we have

$$g_n \overset{\text{def.}}{=} |f_n - f| \in \mathcal{L}^1 \cap \mathcal{M}_+ \quad \wedge \quad \lim_{n\to\infty} g_n = 0.$$

Moreover, from **the triangle inequality for the absolute value**, and **ordered field properties of $\mathbb{R}$**, we have $g_n \leqslant |f_n| + |-f| \leqslant 2g$. Thus, from Lemma 888 ($\mathcal{L}^1$ *is seminormed vector space*), Definition 237 (*seminorm*), Definition 61 (*vector space*, $(\mathcal{L}^1, +)$ is an abelian group), Definition 884 ($\mathcal{L}^1$, *vector space of integrable functions*, $\mathcal{L}^1 \subset \mathcal{M}$), and Definition 593 ($\mathcal{M}_+$, *subset of nonnegative measurable numeric functions*), we have $2g - g_n \in \mathcal{L}^1 \cap \mathcal{M}_+$, and from Lemma 391 (*limit inferior and limit superior of pointwise convergent*), we have $\liminf_{n\to\infty}(2g - g_n) = \lim_{n\to\infty}(2g - g_n) = 2g$.

From Theorem 817 (*Fatou's lemma*, with $f_n \stackrel{\text{def.}}{=} 2g - g_n$), Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*, linearity), and Lemma 384 (*duality limit inferior-limit superior*), we have

$$2\int g\,d\mu = \int \liminf_{n\to\infty}(2g-g_n)\,d\mu \leqslant \liminf_{n\to\infty}\int(2g-g_n)\,d\mu$$
$$= \liminf_{n\to\infty}\left(2\int g\,d\mu - \int g_n\,d\mu\right) = 2\int g\,d\mu - \limsup_{n\to\infty}\int g_n\,d\mu.$$

Thus, from **ordered field properties of $\mathbb{R}$ (with $\int g\,d\mu$ finite)**, $\limsup_{n\to\infty}\int g_n\,d\mu \leqslant 0$. Hence, from Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*, $\int g_n\,d\mu \geqslant 0$), Lemma 392 (*limit inferior bounded from below*, $\liminf_{n\to\infty}\int g_n\,d\mu \geqslant 0$), Lemma 396 (*limit inferior, limit superior and pointwise convergence*, with $\limsup \leqslant 0 \leqslant \liminf$), the definition of the g_n's, and Lemma 874 (*seminorm $\mathcal{L}^1$*), we have

$$0 = \liminf_{n\to\infty}\int g_n\,d\mu = \limsup_{n\to\infty}\int g_n\,d\mu = \lim_{n\to\infty}\int g_n\,d\mu = \lim_{n\to\infty} N_1(f_n - f).$$

Therefore, from Definition 889 (*convergence in $\mathcal{L}^1$*), $(f_n)_{n\in\mathbb{N}}$ is convergent towards f in $\mathcal{L}^1$.

Moreover, from Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*), Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*), and the definition of the g_n's, we have

$$0 \leqslant \left|\int f_n\,d\mu - \int f\,d\mu\right| = \left|\int (f_n - f)\,d\mu\right| \leqslant \int |f_n - f|\,d\mu = \int g_n\,d\mu.$$

Thus, from **the squeeze theorem**, and **linearity of the limit**, we have

$$\lim_{n\to\infty}\int f_n\,d\mu = \int f\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu.$$

□

Remark 898. See the sketch of next proof in Section 5.1.

Theorem 899 *(Lebesgue, extended dominated convergence).*
Let (X, Σ, μ) be a measure space. Let $(f_n)_{n\in\mathbb{N}}, f, g \in \mathcal{M}$. Assume that the sequence is μ-almost everywhere pointwise convergent towards f, that g is μ-integrable, and that for all $n \in \mathbb{N}$, we have $|f_n| \stackrel{\mu\text{ a.e.}}{\leqslant} g$. Then, for all $n \in \mathbb{N}$, f_n is μ-integrable, f is μ-integrable, the sequence $(N_1(f_n - f))_{n\in\mathbb{N}}$ converges towards 0, and we have

$$\int f\,d\mu = \lim_{n\to\infty}\int f_n\,d\mu. \tag{14.23}$$

Proof. For all $n \in \mathbb{N}$, let $\widetilde{B} \stackrel{\text{def.}}{=} \{f = \liminf_{n\to\infty} f_n\} \cap \{f = \limsup_{n\to\infty} f_n\}$ and $\widetilde{C}_n \stackrel{\text{def.}}{=} \{|f_n| \leqslant g\}$.

Let $n \in \mathbb{N}$. Then, from Definition 641 (*property almost satisfied*), Definition 631 (*negligible subset*), Definition 611 (*measure*), Definition 516 (*measurable space*, Σ is a σ-algebra), Definition 474 (*σ-algebra*, closedness under complement), **monotonicity of complement**, and since **involutiveness of complement**, let $B, C_n \in \Sigma$ such that $B \subset \widetilde{B}$, $C_n \subset \widetilde{C}_n$ and $\mu(B^c) = \mu(C_n^c) = 0$.

Let $C \stackrel{\text{def.}}{=} \bigcap_{n\in\mathbb{N}} C_n$ and $D \stackrel{\text{def.}}{=} B \cap C$. Then, from Lemma 475 (*equivalent definition of σ-algebra*, closedness under countable intersection (with $I = \mathbb{N}$, then $\operatorname{card}(I) = 2$)), **De Morgan's laws**, and Lemma 638 (*compatibility of null measure with countable union*, with $I = \mathbb{N}$, then $\operatorname{card}(I) = 2$), we have $C, D \in \Sigma$, and $\mu(C^c) = \mu(D^c) = 0$. For all $n \in \mathbb{N}$, let

$$A \stackrel{\text{def.}}{=} D \cap g^{-1}(\mathbb{R}_+), \quad \tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbb{1}_A, \quad \tilde{f} \stackrel{\text{def.}}{=} f \mathbb{1}_A, \quad \tilde{g} \stackrel{\text{def.}}{=} g \mathbb{1}_A.$$

Let $n \in \mathbb{N}$. Then, from Lemma 688 (*finite nonnegative part*, with g), we have $A \in \Sigma$, $\tilde{g}$ belongs to $\mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$, $\mu(A^c) = 0$, and $g \stackrel{\mu\text{ a.e.}}{=} \tilde{g}$. Hence, from Lemma 591 (*measurability and masking*, with f_n and f), and Lemma 687 (*masking almost nowhere*, with f_n and f), we have $\tilde{f}_n, \tilde{f} \in \mathcal{M}$, $f_n \overset{\mu\,\text{a.e.}}{=} \tilde{f}_n$ and $f \overset{\mu\,\text{a.e.}}{=} \tilde{f}$.

From Lemma 660 (*compatibility of almost equality with operator*, with the unary operator absolute value), we have $|\tilde{g}| \overset{\mu\,\text{a.e.}}{=} |g|$. Thus, from Lemma 862 (*compatibility of integral with almost equality*, with $|g|$ and $|\tilde{g}|$), Lemma 853 (*equivalent definition of integrability*, with g), and Lemma 874 (*seminorm $\mathcal{L}^1$*), we have $N_1(\tilde{g}) = N_1(g) < \infty$. Hence, from Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), we have $\tilde{g} \in \mathcal{L}^1$.

Let $x \in X$. **Case $x \in A$.** Then, from **the definition of the indicator function**, and since $A \subset B \subset \widetilde{B}$, we have $\lim_{n\to\infty} \tilde{f}_n(x) = \lim_{n\to\infty} f_n(x) = f(x) = \tilde{f}(x)$. Moreover, for all $n \in \mathbb{N}$, since $A \subset C_n \subset \widetilde{C}_n$, we have $|\tilde{f}_n(x)| = |f_n(x)| \leqslant g(x) = \tilde{g}(x)$. **Case $x \notin A$.** Then, from **the definition of the indicator function**, we have for all $n \in \mathbb{N}$, $\tilde{f}_n(x) = \tilde{f}(x) = 0 = \tilde{g}(x)$. Thus, we have $\lim_{n\to\infty} \tilde{f}_n(x) = 0 = \tilde{f}(x)$, and $|\tilde{f}_n(x)| = 0 \leqslant 0 = \tilde{g}(x)$. Hence, in all cases, from Theorem 897 (*Lebesgue, dominated convergence*), we have for all $n \in \mathbb{N}$, $\tilde{f}_n, \tilde{f} \in \mathcal{L}^1$, and

$$\int \tilde{f}\,d\mu = \lim_{n\to\infty} \int \tilde{f}_n\,d\mu.$$

Therefore, from Lemma 660 (*compatibility of almost equality with operator*, with the unary operator absolute value, $|\tilde{f}_n| \overset{\mu\,\text{a.e.}}{=} |f_n|$ and $|\tilde{f}| \overset{\mu\,\text{a.e.}}{=} |f|$), Lemma 862 (*compatibility of integral with almost equality*, with f_n and $\tilde{f}_n$, then f and $\tilde{f}$), Lemma 853 (*equivalent definition of integrability*, with f_n, then f), Lemma 874 (*seminorm $\mathcal{L}^1$*), and Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*), for all $n \in \mathbb{N}$, f_n and f are μ-integrable in $\mathcal{M}$, and

$$\int f\,d\mu = \int \tilde{f}\,d\mu = \lim_{n\to\infty} \int \tilde{f}_n\,d\mu. = \lim_{n\to\infty} \int f\,d\mu.$$

□

Chapter 15

Conclusions, perspectives

We have presented very detailed proofs of the main basic results in measure theory and Lebesgue integration such as the Beppo Levi (monotone convergence) theorem, Fatou's lemma, the Tonelli theorem, and Lebesgue's dominated convergence theorem.

The short-term purpose of this work was to help the formalization in a formal proof assistant such as Coq of the basic concepts in measure theory and Lebesgue integration and of the proofs of their main properties. A first milestone towards this is dedicated to the integral of nonnegative measurable functions [8] where special attention was paid to the formalization of σ-algebras and of simple functions.

Our mean-term purpose is now to continue up to the formalization of L^p Lebesgue spaces and of $W^{m,p}$ Sobolev spaces as Banach spaces, and in particular L^2 and $H^m \overset{\text{def.}}{=} W^{m,2}$ as Hilbert spaces. This will include parts of the distribution theory.

The long-term purpose of these studies is the formal proof of programs implementing the Finite Element Method. As a consequence, after having addressed the formalization of the Lax–Milgram theorem [17, 7], we will also have to write very detailed pen-and-paper proofs for the concepts and results of the interpolation and approximation theory to define the Finite Element Method itself.

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Appendix A

Lists of statements

This appendix collects the references (name and number) for all statements present in Part II. These are split into definitions, lemmas, and theorems.

List of Definitions

List of Lemmas

List of Theorems

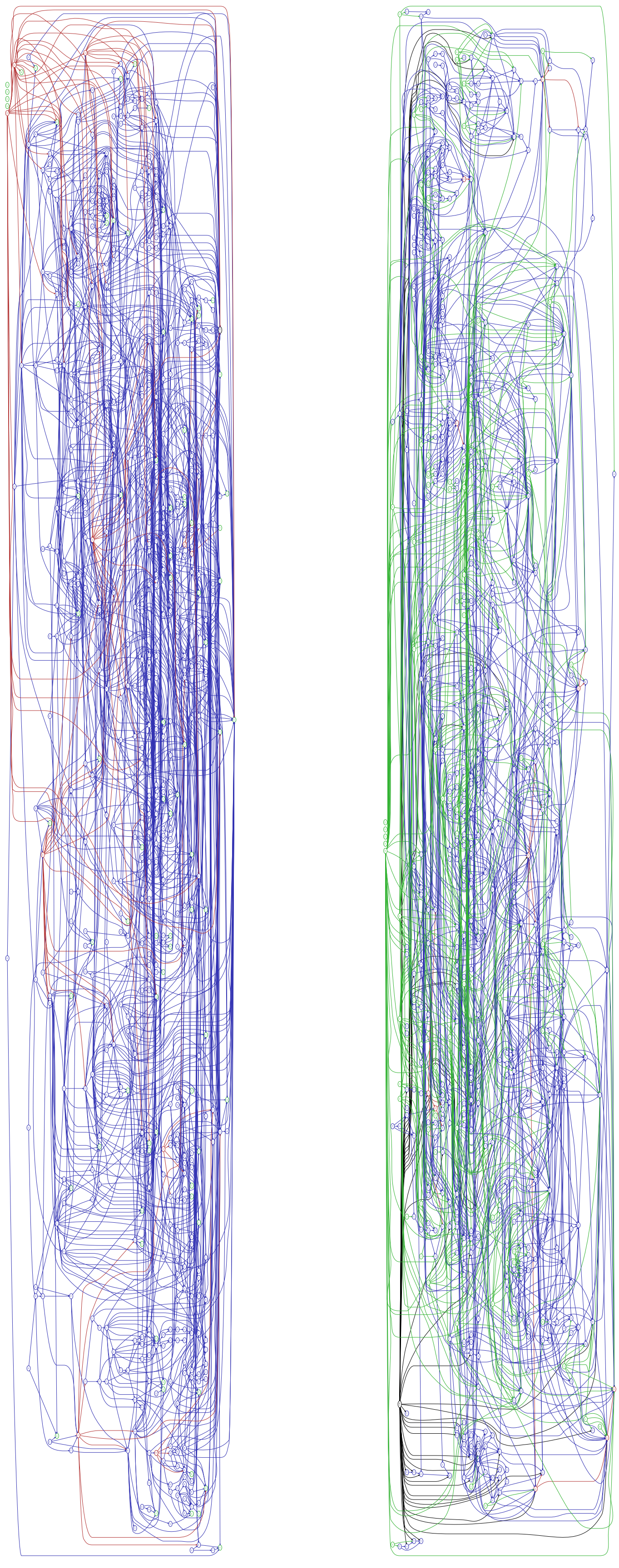

Figure A.1: Dependency graphs. Top: the proof cites explicitly… Bottom: is explicitly cited in the proof of… Arrows are colored according to the nature of their foot: green is for definitions, blue for lemmas, and red for theorems. All dependencies are listed in Appendices B and C.

Appendix B

The proof cites explicitly...

This appendix gathers the explicit citations of the statements listed in Appendix A that appear in the proof of each result (lemmas and theorems). Statements from [17] are anonymized.

The corresponding dependency graph is represented in Figure A.1 (top). The dual graph is described in Appendix C.

Printing is not advised!

The proof of Lemma 209 (*compatibility of pseudopartition with intersection*)
cites explicitly:
Definition 207 (*pseudopartition*).

The proof of Lemma 210 (*technical inclusion for countable union*)
has no explicit citation.

The proof of Lemma 211 (*order is meaningless in countable union*)
cites explicitly:
Lemma 210 (*technical inclusion for countable union*).

The proof of Lemma 212 (*definition of double countable union*)
cites explicitly:
Lemma 211 (*order is meaningless in countable union*).

The proof of Lemma 213 (*double countable union*)
cites explicitly:
Lemma 212 (*definition of double countable union*).

The proof of Lemma 215 (*partition of countable union*)
has no explicit citation.

The proof of Lemma 218 (*restriction is masking*)
has no explicit citation.

The proof of Lemma 220 (*quotient vector operations*)
cites explicitly:
Definition 219 (*relation compatible with vector operations*).

The proof of Lemma 221 (*quotient vector space, equivalence relation*)
cites explicitly:
Statement(s) from [17],
Lemma 220 (*quotient vector operations*).

The proof of Lemma 222 (*quotient vector space*)
cites explicitly:
Statement(s) from [17],
Definition 219 (*relation compatible with vector operations*),
Lemma 221 (*quotient vector space, equivalence relation*).

The proof of Lemma 223 (*linear map on quotient vector space*)
cites explicitly:
Statement(s) from [17],
Lemma 220 (*quotient vector operations*),
Lemma 222 (*quotient vector space*).

The proof of Lemma 228 ($\mathbb{K}$ *is* $\mathbb{K}$*-algebra*)
cites explicitly:
Definition 226 (*algebra over a field*).

The proof of Lemma 231 (*algebra of functions to algebra*)
cites explicitly:
Statement(s) from [17],
Definition 226 (*algebra over a field*),
Definition 229 (*inherited algebra operations*).

The proof of Lemma 232 ($\mathbb{K}^X$ *is algebra*)
cites explicitly:
Lemma 228 ($\mathbb{K}$ *is* $\mathbb{K}$*-algebra*),
Lemma 231 (*algebra of functions to algebra*).

The proof of Lemma 235 (*vector subspace and closed under multiplication is subalgebra*)
cites explicitly:
Statement(s) from [17],
Definition 226 (*algebra over a field*),
Definition 233 (*subalgebra*).

The proof of Lemma 236 (*closed under algebra operations is subalgebra*)
cites explicitly:
Statement(s) from [17],
Lemma 235 (*vector subspace and closed under multiplication is subalgebra*).

The proof of Lemma 239 (*definite seminorm is norm*)
cites explicitly:
Statement(s) from [17],
Definition 237 (*seminorm*).

The proof of Lemma 243 (*empty open interval*)
cites explicitly:
Definition 241 (*interval*).

The proof of Lemma 246 (*intervals are closed under finite intersection*)
cites explicitly:
Definition 241 (*interval*).

The proof of Lemma 247 (*empty intersection of open intervals*)
cites explicitly:
Lemma 243 (*empty open interval*),
Lemma 246 (*intervals are closed under finite intersection*).

The proof of Lemma 250 (*intersection of topologies*)
cites explicitly:
Definition 249 (*topological space*).

The proof of Lemma 252 (*generated topology is minimum*)
cites explicitly:
Lemma 250 (*intersection of topologies*),
Definition 251 (*generated topology*).

The proof of Lemma 253 (*equivalent definition of generated topology*)
cites explicitly:
Definition 249 (*topological space*),
Lemma 252 (*generated topology is minimum*).

The proof of Lemma 255 (*augmented topological basis*)
cites explicitly:
Definition 254 (*topological basis*).

The proof of Lemma 258 (*topological basis of order topology*)
cites explicitly:
Lemma 246 (*intervals are closed under finite intersection*),
Lemma 253 (*equivalent definition of generated topology*),
Definition 254 (*topological basis*),
Definition 256 (*order topology*).

The proof of Lemma 260 (*trace topology on subset*)
cites explicitly:
Definition 216 (*trace of subsets of parties*),
Definition 249 (*topological space*),
Definition 254 (*topological basis*).

The proof of Lemma 261 (*box topology on Cartesian product*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Definition 249 (*topological space*),
Definition 254 (*topological basis*).

The proof of Lemma 264 (*complete countable topological basis*)
cites explicitly:
Lemma 255 (*augmented topological basis*),
Definition 262 (*second-countability*).

The proof of Lemma 265 (*compatibility of second-countability with Cartesian product*)

cites explicitly:
Definition 217 (*product of subsets of parties*),
Lemma 261 (*box topology on Cartesian product*).

The proof of Lemma 266 (*complete countable topological basis of product space*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Lemma 261 (*box topology on Cartesian product*),
Lemma 264 (*complete countable topological basis*).

The proof of Lemma 269 (*equivalent definition of convergent sequence*)
cites explicitly:
Statement(s) from [17].

The proof of Lemma 270 (*convergent subsequence of Cauchy sequence*)
cites explicitly:
Statement(s) from [17].

The proof of Lemma 272 (*finite cover of compact interval*)
has no explicit citation.

The proof of Lemma 274 (*2 is self-Hölder conjugate in* $\mathbb{R}$)
cites explicitly:
Definition 273 (*Hölder conjugates in* $\mathbb{R}$).

The proof of Lemma 275 (*Young's inequality for products in* $\mathbb{R}$)
cites explicitly:
Definition 273 (*Hölder conjugates in* $\mathbb{R}$).

The proof of Lemma 276 (*Young's inequality for products in* $\mathbb{R}$*, case* $p = 2$)
cites explicitly:
Lemma 274 (*2 is self-Hölder conjugate in* $\mathbb{R}$),
Lemma 275 (*Young's inequality for products in* $\mathbb{R}$).

The proof of Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*)
cites explicitly:
Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$).

The proof of Lemma 283 (*zero is identity element for addition in* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 284 (*addition in* $\overline{\mathbb{R}}$ *is associative when defined*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 285 (*addition in* $\overline{\mathbb{R}}$ *is commutative when defined*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 286 (*infinity-sum property in* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 287 (*additive inverse in* $\overline{\mathbb{R}}$ *is monotone*)
cites explicitly:
Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$),
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 290 (*multiplication in* $\overline{\mathbb{R}}$ *is associative when defined*)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$).

The proof of Lemma 291 (*multiplication in* $\overline{\mathbb{R}}$ *is commutative when defined*)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$).

The proof of Lemma 292 (*multiplication in* $\overline{\mathbb{R}}$ *is left distributive over addition when defined*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$).

The proof of Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*)
cites explicitly:
Lemma 291 (*multiplication in $\overline{\mathbb{R}}$ is commutative when defined*),
Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*).

The proof of Lemma 294 (*zero-product property in $\overline{\mathbb{R}}$*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*).

The proof of Lemma 295 (*infinity-product property in $\overline{\mathbb{R}}$*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*).

The proof of Lemma 296 (*finite-product property in $\overline{\mathbb{R}}$*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*).

The proof of Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*)
cites explicitly:
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 299 (*bounded absolute value in $\overline{\mathbb{R}}$*)
cites explicitly:
Lemma 287 (*additive inverse in $\overline{\mathbb{R}}$ is monotone*),
Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 300 (*bounded absolute value in $\overline{\mathbb{R}}$ (strict)*)
cites explicitly:
Lemma 287 (*additive inverse in $\overline{\mathbb{R}}$ is monotone*),
Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 301 (*finite absolute value in $\overline{\mathbb{R}}$*)
cites explicitly:
Lemma 300 (*bounded absolute value in $\overline{\mathbb{R}}$ (strict)*).

The proof of Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*)
cites explicitly:
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*)
cites explicitly:
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*)
cites explicitly:
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*)
cites explicitly:
Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*),
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*).

The proof of Lemma 307 (*exponential and logarithm in $\overline{\mathbb{R}}$ are inverse*)
cites explicitly:
Definition 306 (*exponential and logarithm in $\overline{\mathbb{R}}$*).

The proof of Lemma 309 (*exponentiation in* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$),
Definition 306 (*exponential and logarithm in* $\overline{\mathbb{R}}$),
Definition 308 (*exponentiation in* $\overline{\mathbb{R}}$).

The proof of Lemma 310 (*topology of* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 256 (*order topology*),
Lemma 258 (*topological basis of order topology*),
Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$).

The proof of Lemma 312 (*trace topology on* $\mathbb{R}$)
cites explicitly:
Definition 254 (*topological basis*),
Lemma 258 (*topological basis of order topology*),
Lemma 260 (*trace topology on subset*).

The proof of Lemma 314 (*convergence towards* $-\infty$)
cites explicitly:
Lemma 310 (*topology of* $\overline{\mathbb{R}}$).

The proof of Lemma 315 (*continuity of addition in* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 316 (*continuity of multiplication in* $\overline{\mathbb{R}}$)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$).

The proof of Lemma 317 (*absolute value in* $\overline{\mathbb{R}}$ *is continuous*)
cites explicitly:
Definition 297 (*absolute value in* $\overline{\mathbb{R}}$).

The proof of Lemma 318 (*addition in* $\overline{\mathbb{R}}_+$ *is closed*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$).

The proof of Lemma 319 (*addition in* $\overline{\mathbb{R}}_+$ *is associative*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Lemma 284 (*addition in* $\overline{\mathbb{R}}$ *is associative when defined*).

The proof of Lemma 320 (*addition in* $\overline{\mathbb{R}}_+$ *is commutative*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Lemma 285 (*addition in* $\overline{\mathbb{R}}$ *is commutative when defined*).

The proof of Lemma 321 (*infinity-sum property in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 286 (*infinity-sum property in* $\overline{\mathbb{R}}$).

The proof of Lemma 322 (*series are convergent in* $\overline{\mathbb{R}}_+$)
has no explicit citation.

The proof of Lemma 323 (*technical upper bound in series in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 322 (*series are convergent in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 324 (*order is meaningless in series in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 323 (*technical upper bound in series in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 325 (*definition of double series in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 324 (*order is meaningless in series in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 326 (*double series in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 322 (*series are convergent in* $\overline{\mathbb{R}}_+$),
Lemma 325 (*definition of double series in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 329 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed when defined*)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$),
Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 330 (*zero-product property in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 294 (*zero-product property in* $\overline{\mathbb{R}}$),
Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 331 (*infinity-product property in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 295 (*infinity-product property in* $\overline{\mathbb{R}}$),
Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 332 (*finite-product property in* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 296 (*finite-product property in* $\overline{\mathbb{R}}$),
Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$).

The proof of Lemma 335 (*zero-product property in* $\overline{\mathbb{R}}$ *(measure theory)*)
cites explicitly:
Lemma 294 (*zero-product property in* $\overline{\mathbb{R}}$),
Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*).

The proof of Lemma 336 (*infinity-product property in* $\overline{\mathbb{R}}$ *(measure theory)*)
cites explicitly:
Lemma 295 (*infinity-product property in* $\overline{\mathbb{R}}$),
Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*).

The proof of Lemma 337 (*finite-product property in* $\overline{\mathbb{R}}$ *(measure theory)*)
cites explicitly:
Lemma 336 (*infinity-product property in* $\overline{\mathbb{R}}$ *(measure theory)*).

The proof of Lemma 338 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed (measure theory)*)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$),
Definition 327 (*multiplication in* $\overline{\mathbb{R}}_+$),
Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*).

The proof of Lemma 340 (*multiplication in* $\overline{\mathbb{R}}_+$ *is associative (measure theory)*)
cites explicitly:
Lemma 290 (*multiplication in* $\overline{\mathbb{R}}$ *is associative when defined*),
Definition 333 (*multiplication in* $\overline{\mathbb{R}}$ *(measure theory)*).

The proof of Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*)
cites explicitly:
Lemma 291 (*multiplication in $\overline{\mathbb{R}}$ is commutative when defined*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*)
cites explicitly:
Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*),
Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
cites explicitly:
Lemma 330 (*zero-product property in $\overline{\mathbb{R}}_+$*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
cites explicitly:
Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 345 (*finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
cites explicitly:
Lemma 337 (*finite-product property in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 346 (*exponentiation in $\overline{\mathbb{R}}$ (measure theory)*)
cites explicitly:
Definition 308 (*exponentiation in $\overline{\mathbb{R}}$*),
Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*).

The proof of Lemma 349 (*Young's inequality for products (measure theory)*)
cites explicitly:
Lemma 275 (*Young's inequality for products in $\mathbb{R}$*),
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*),
Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*),
Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*),
Lemma 329 (*multiplication in $\overline{\mathbb{R}}_+$ is closed when defined*),
Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*),
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*).

The proof of Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*)
cites explicitly:
Lemma 274 (*2 is self-Hölder conjugate in $\mathbb{R}$*),
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*),
Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*),
Lemma 349 (*Young's inequality for products (measure theory)*).

The proof of Lemma 352 (*connected component of open subset of $\mathbb{R}$ is open interval*)

cites explicitly:

Statement(s) from [17],
Definition 241 (*interval*),
Definition 249 (*topological space*),
Definition 351 (*connected component in* $\mathbb{R}$).

The proof of Lemma 353 (*connected component of open subset of* $\mathbb{R}$ *is maximal*)
cites explicitly:
Definition 351 (*connected component in* $\mathbb{R}$),
Lemma 352 (*connected component of open subset of* $\mathbb{R}$ *is open interval*).

The proof of Lemma 354 (*connected components of open subset of* $\mathbb{R}$ *are equal or disjoint*)
cites explicitly:
Lemma 353 (*connected component of open subset of* $\mathbb{R}$ *is maximal*).

The proof of Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$)
cites explicitly:
Statement(s) from [17],
Definition 351 (*connected component in* $\mathbb{R}$),
Lemma 354 (*connected components of open subset of* $\mathbb{R}$ *are equal or disjoint*).

The proof of Lemma 356 (*rational approximation of lower bound of open interval*)
has no explicit citation.

The proof of Lemma 357 (*rational approximation of upper bound of open interval*)
cites explicitly:
Lemma 356 (*rational approximation of lower bound of open interval*).

The proof of Lemma 358 (*open intervals with rational bounds cover open interval*)
cites explicitly:
Statement(s) from [17],
Definition 249 (*topological space*),
Lemma 356 (*rational approximation of lower bound of open interval*),
Lemma 357 (*rational approximation of upper bound of open interval*).

The proof of Theorem 359 ($\mathbb{R}$ *is second-countable*)
cites explicitly:
Definition 254 (*topological basis*),
Definition 262 (*second-countability*),
Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$),
Lemma 358 (*open intervals with rational bounds cover open interval*).

The proof of Lemma 360 ($\mathbb{R}^n$ *is second-countable*)
cites explicitly:
Lemma 261 (*box topology on Cartesian product*),
Definition 262 (*second-countability*),
Lemma 265 (*compatibility of second-countability with Cartesian product*),
Theorem 359 ($\mathbb{R}$ *is second-countable*).

The proof of Lemma 361 (*open intervals with rational bounds cover open interval of* $\overline{\mathbb{R}}$)
cites explicitly:
Lemma 356 (*rational approximation of lower bound of open interval*),
Lemma 357 (*rational approximation of upper bound of open interval*),
Lemma 358 (*open intervals with rational bounds cover open interval*).

The proof of Lemma 362 (*$\overline{\mathbb{R}}$ is second-countable*)
cites explicitly:
Definition 254 (*topological basis*),
Definition 262 (*second-countability*),
Lemma 310 (*topology of $\overline{\mathbb{R}}$*),
Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*),
Lemma 361 (*open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$*).

The proof of Lemma 363 (*extrema of constant function*)
cites explicitly:
Statement(s) from [17].

The proof of Lemma 364 (*equivalent definition of finite infimum*)
cites explicitly:
Statement(s) from [17],
Lemma 269 (*equivalent definition of convergent sequence*).

The proof of Lemma 365 (*equivalent definition of finite infimum in $\overline{\mathbb{R}}$*)
cites explicitly:
Statement(s) from [17],
Lemma 363 (*extrema of constant function*),
Lemma 364 (*equivalent definition of finite infimum*).

The proof of Lemma 366 (*equivalent definition of infimum*)
cites explicitly:
Statement(s) from [17],
Lemma 314 (*convergence towards $-\infty$*),
Lemma 365 (*equivalent definition of finite infimum in $\overline{\mathbb{R}}$*).

The proof of Lemma 367 (*infimum is smaller than supremum*)
cites explicitly:
Statement(s) from [17],
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*).

The proof of Lemma 368 (*infimum is monotone*)
cites explicitly:
Statement(s) from [17],
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*).

The proof of Lemma 369 (*supremum is monotone*)
cites explicitly:
Statement(s) from [17],
Lemma 368 (*infimum is monotone*).

The proof of Lemma 370 (*compatibility of infimum with absolute value*)
cites explicitly:
Statement(s) from [17],
Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*),
Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*),
Lemma 367 (*infimum is smaller than supremum*),
Lemma 368 (*infimum is monotone*),
Lemma 369 (*supremum is monotone*).

The proof of Lemma 371 (*compatibility of supremum with absolute value*)
cites explicitly:
Statement(s) from [17],
Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*),
Lemma 370 (*compatibility of infimum with absolute value*).

The proof of Lemma 372 (*compatibility of translation with infimum*)
cites explicitly:
Lemma 368 (*infimum is monotone*).

The proof of Lemma 373 (*compatibility of translation with supremum*)
cites explicitly:
Lemma 369 (*supremum is monotone*).

The proof of Lemma 374 (*infimum of sequence is monotone*)
cites explicitly:
Statement(s) from [17],
Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*).

The proof of Lemma 375 (*supremum of sequence is monotone*)
cites explicitly:
Statement(s) from [17],
Lemma 374 (*infimum of sequence is monotone*).

The proof of Lemma 376 (*infimum of bounded sequence is bounded*)
cites explicitly:
Lemma 363 (*extrema of constant function*),
Lemma 374 (*infimum of sequence is monotone*).

The proof of Lemma 377 (*supremum of bounded sequence is bounded*)
cites explicitly:
Lemma 363 (*extrema of constant function*),
Lemma 375 (*supremum of sequence is monotone*).

The proof of Lemma 378 (*limit inferior*)
cites explicitly:
Lemma 368 (*infimum is monotone*).

The proof of Lemma 379 (*limit inferior is* ∞)
cites explicitly:
Statement(s) from [17],
Lemma 378 (*limit inferior*).

The proof of Lemma 380 (*equivalent definition of the limit inferior*)
cites explicitly:
Statement(s) from [17],
Definition 271 (*cluster point*),
Lemma 378 (*limit inferior*),
Lemma 379 (*limit inferior is* ∞).

The proof of Lemma 381 (*limit inferior is invariant by translation*)
cites explicitly:
Definition 271 (*cluster point*),
Lemma 380 (*equivalent definition of the limit inferior*).

The proof of Lemma 382 (*limit inferior is monotone*)
cites explicitly:
Lemma 374 (*infimum of sequence is monotone*),
Lemma 375 (*supremum of sequence is monotone*),
Lemma 378 (*limit inferior*),
Lemma 381 (*limit inferior is invariant by translation*).

The proof of Lemma 383 (*limit superior*)
has no explicit citation.

The proof of Lemma 384 (*duality limit inferior-limit superior*)
cites explicitly:
Statement(s) from [17],
Lemma 378 (*limit inferior*),
Lemma 383 (*limit superior*).

The proof of Lemma 385 (*equivalent definition of limit superior*)
cites explicitly:
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 384 (*duality limit inferior-limit superior*).

The proof of Lemma 386 (*limit inferior is smaller than limit superior*)
cites explicitly:
Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*),
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 385 (*equivalent definition of limit superior*).

The proof of Lemma 387 (*limit superior is monotone*)
cites explicitly:
Lemma 382 (*limit inferior is monotone*),
Lemma 384 (*duality limit inferior-limit superior*).

The proof of Lemma 388 (*compatibility of limit inferior with absolute value*)
cites explicitly:
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 378 (*limit inferior*),
Lemma 383 (*limit superior*).

The proof of Lemma 389 (*compatibility of limit superior with absolute value*)
cites explicitly:
Lemma 303 (*absolute value in* $\overline{\mathbb{R}}$ *is even*),
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 388 (*compatibility of limit inferior with absolute value*).

The proof of Lemma 391 (*limit inferior and limit superior of pointwise convergent*)
cites explicitly:
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 385 (*equivalent definition of limit superior*),
Definition 390 (*pointwise convergence*).

The proof of Lemma 392 (*limit inferior bounded from below*)
cites explicitly:
Statement(s) from [17],
Lemma 382 (*limit inferior is monotone*),
Lemma 391 (*limit inferior and limit superior of pointwise convergent*).

The proof of Lemma 393 (*limit inferior bounded from above*)
cites explicitly:
Statement(s) from [17],
Lemma 382 (*limit inferior is monotone*),
Lemma 391 (*limit inferior and limit superior of pointwise convergent*).

The proof of Lemma 394 (*limit superior bounded from below*)
cites explicitly:
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 393 (*limit inferior bounded from above*).

The proof of Lemma 395 (*limit superior bounded from above*)
cites explicitly:
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 392 (*limit inferior bounded from below*).

The proof of Lemma 396 (*limit inferior, limit superior and pointwise convergence*)
cites explicitly:
Statement(s) from [17],
Lemma 379 (*limit inferior is* ∞),
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 385 (*equivalent definition of limit superior*),
Lemma 386 (*limit inferior is smaller than limit superior*).

The proof of Lemma 398 (*finite part is finite*)
cites explicitly:
Definition 278 (*extended real numbers,* $\overline{\mathbb{R}}$),
Definition 397 (*finite part*).

The proof of Lemma 400 (*equivalent definition of nonnegative and nonpositive parts*)

cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*).

The proof of Lemma 401 (*nonnegative and nonpositive parts are nonnegative*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*).

The proof of Lemma 402 (*nonnegative and nonpositive parts are orthogonal*)
has no explicit citation.

The proof of Lemma 403 (*decomposition into nonnegative and nonpositive parts*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Definition 297 (*absolute value in* $\overline{\mathbb{R}}$),
Lemma 402 (*nonnegative and nonpositive parts are orthogonal*).

The proof of Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Lemma 319 (*addition in* $\overline{\mathbb{R}}_+$ *is associative*),
Lemma 320 (*addition in* $\overline{\mathbb{R}}_+$ *is commutative*),
Lemma 321 (*infinity-sum property in* $\overline{\mathbb{R}}_+$),
Definition 399 (*nonnegative and nonpositive parts*),
Lemma 401 (*nonnegative and nonpositive parts are nonnegative*),
Lemma 402 (*nonnegative and nonpositive parts are orthogonal*),
Lemma 403 (*decomposition into nonnegative and nonpositive parts*).

The proof of Lemma 405 (*compatibility of nonpositive and nonnegative parts with mask*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*).

The proof of Lemma 406 (*compatibility of nonpositive and nonnegative parts with restriction*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*).

The proof of Lemma 408 (*nonempty and with empty or full*)
has no explicit citation.

The proof of Lemma 409 (*with empty and full*)
has no explicit citation.

The proof of Lemma 411 (*closedness under local complement and complement*)
has no explicit citation.

The proof of Lemma 412 (*closedness under disjoint union and local complement*)
has no explicit citation.

The proof of Lemma 413 (*closedness under set difference and local complement*)
has no explicit citation.

The proof of Lemma 414 (*closedness under intersection and set difference*)
has no explicit citation.

The proof of Lemma 415 (*closedness under union and intersection*)
has no explicit citation.

The proof of Lemma 416 (*closedness under union and set difference*)
cites explicitly:
Lemma 414 (*closedness under intersection and set difference*),
Lemma 415 (*closedness under union and intersection*).

The proof of Lemma 417 (*closedness under finite operations*)
has no explicit citation.

The proof of Lemma 418 (*closedness under finite union and intersection*)
cites explicitly:
Lemma 415 (*closedness under union and intersection*),
Lemma 417 (*closedness under finite operations*).

The proof of Lemma 421 (*closedness under countable and finite disjoint union*)
has no explicit citation.

The proof of Lemma 422 (*closedness under countable disjoint union and local complement*)
cites explicitly:
Lemma 409 (*with empty and full*),
Lemma 412 (*closedness under disjoint union and local complement*),
Lemma 417 (*closedness under finite operations*),
Lemma 421 (*closedness under countable and finite disjoint union*).

The proof of Lemma 423 (*closedness under countable union and intersection*)
has no explicit citation.

The proof of Lemma 424 (*closedness under countable disjoint and monotone union*)
cites explicitly:
Lemma 215 (*partition of countable union*).

The proof of Lemma 425 (*closedness under countable monotone and disjoint union*)
has no explicit citation.

The proof of Lemma 426 (*closedness under countable disjoint union and countable union*)
cites explicitly:
Lemma 215 (*partition of countable union*),

Lemma 414 (*closedness under intersection and set difference*),
Lemma 415 (*closedness under union and intersection*),
Lemma 417 (*closedness under finite operations*).

The proof of Lemma 427 (*closedness under countable monotone union and countable union*)
cites explicitly:
Lemma 413 (*closedness under set difference and local complement*),
Lemma 414 (*closedness under intersection and set difference*),
Lemma 415 (*closedness under union and intersection*),
Lemma 425 (*closedness under countable monotone and disjoint union*),
Lemma 426 (*closedness under countable disjoint union and countable union*).

The proof of Lemma 431 (*intersection of π-systems*)
cites explicitly:
Definition 428 (*π-system*).

The proof of Lemma 433 (*generated π-system is minimum*)
cites explicitly:
Lemma 431 (*intersection of π-systems*),
Definition 432 (*generated π-system*).

The proof of Lemma 434 (*π-system generation is monotone*)
cites explicitly:
Lemma 433 (*generated π-system is minimum*).

The proof of Lemma 435 (*π-system generation is idempotent*)
cites explicitly:
Lemma 433 (*generated π-system is minimum*).

The proof of Lemma 438 (*equivalent definition of set algebra*)
cites explicitly:
Lemma 408 (*nonempty and with empty or full*),
Lemma 409 (*with empty and full*),
Lemma 417 (*closedness under finite operations*),
Lemma 418 (*closedness under finite union and intersection*),
Definition 437 (*set algebra*).

The proof of Lemma 439 (*other equivalent definition of set algebra*)
cites explicitly:
Lemma 411 (*closedness under local complement and complement*),
Lemma 413 (*closedness under set difference and local complement*),
Lemma 414 (*closedness under intersection and set difference*),
Lemma 438 (*equivalent definition of set algebra*).

The proof of Lemma 440 (*set algebra is closed under local complement*)
cites explicitly:
Lemma 413 (*closedness under set difference and local complement*),
Lemma 439 (*other equivalent definition of set algebra*).

The proof of Lemma 441 (*intersection of set algebras*)
cites explicitly:
Definition 437 (*set algebra*).

The proof of Lemma 443 (*generated set algebra is minimum*)
cites explicitly:
Lemma 441 (*intersection of set algebras*),
Definition 442 (*generated set algebra*).

The proof of Lemma 444 (*set algebra generation is monotone*)
cites explicitly:
Lemma 443 (*generated set algebra is minimum*).

The proof of Lemma 445 (*set algebra generation is idempotent*)
cites explicitly:
Lemma 443 (*generated set algebra is minimum*).

The proof of Lemma 446 (*partition of countable union in set algebra*)
cites explicitly:
Lemma 215 (*partition of countable union*),
Lemma 438 (*equivalent definition of set algebra*),
Lemma 439 (*other equivalent definition of set algebra*).

The proof of Lemma 447 (*explicit set algebra*)
cites explicitly:
Lemma 417 (*closedness under finite operations*),
Definition 437 (*set algebra*),
Lemma 438 (*equivalent definition of set algebra*),
Lemma 443 (*generated set algebra is minimum*).

The proof of Lemma 449 (*intersection of monotone classes*)
cites explicitly:
Definition 448 (*monotone class*).

The proof of Lemma 451 (*generated monotone class is minimum*)
cites explicitly:
Lemma 449 (*intersection of monotone classes*),
Definition 450 (*generated monotone class*).

The proof of Lemma 452 (*monotone class generation is monotone*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*).

The proof of Lemma 453 (*monotone class generation is idempotent*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*).

The proof of Lemma 455 ($\mathcal{C}^{\setminus}$ *is symmetric*)
cites explicitly:
Definition 454 (*monotone class and symmetric set difference*).

The proof of Lemma 456 ($\mathcal{C}^{\setminus}$ *is monotone class*)
cites explicitly:
Definition 448 (*monotone class*),
Definition 454 (*monotone class and symmetric set difference*).

The proof of Lemma 457 (*monotone class is closed under set difference*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*),
Definition 454 (*monotone class and symmetric set difference*),
Lemma 455 ($\mathcal{C}^{\setminus}$ *is symmetric*),
Lemma 456 ($\mathcal{C}^{\setminus}$ *is monotone class*).

The proof of Lemma 458 (*monotone class generated by set algebra*)
cites explicitly:
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 451 (*generated monotone class is minimum*),
Lemma 457 (*monotone class is closed under set difference*).

The proof of Lemma 460 (*equivalent definition of λ-system*)
cites explicitly:
Lemma 411 (*closedness under local complement and complement*),
Lemma 422 (*closedness under countable disjoint union and local complement*),
Lemma 424 (*closedness under countable disjoint and monotone union*),
Lemma 425 (*closedness under countable monotone and disjoint union*),
Definition 459 (*λ-system*).

The proof of Lemma 461 (*other properties of λ-system*)
cites explicitly:
Lemma 408 (*nonempty and with empty or full*),
Lemma 409 (*with empty and full*),
Lemma 423 (*closedness under countable union and intersection*),
Definition 459 (*λ-system*),
Lemma 460 (*equivalent definition of λ-system*).

The proof of Lemma 462 (*intersection of λ-systems*)
cites explicitly:
Definition 459 (*λ-system*).

The proof of Lemma 464 (*generated λ-system is minimum*)
cites explicitly:
Lemma 462 (*intersection of λ-systems*),
Definition 463 (*generated λ-system*).

The proof of Lemma 465 (*λ-system generation is monotone*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*).

The proof of Lemma 466 (*λ-system generation is idempotent*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*).

The proof of Lemma 468 (*$\Lambda^\cap$ is symmetric*)
cites explicitly:
Definition 467 (*λ-system and intersection*).

The proof of Lemma 469 (*$\Lambda^\cap$ is λ-system*)
cites explicitly:
Definition 459 (*λ-system*),
Lemma 460 (*equivalent definition of λ-system*).

The proof of Lemma 470 (*λ-system with intersection*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*),
Definition 467 (*λ-system and intersection*),
Lemma 468 (*$\Lambda^\cap$ is symmetric*),
Lemma 469 (*$\Lambda^\cap$ is λ-system*).

The proof of Lemma 471 (*λ-system is closed under intersection*)
cites explicitly:
Definition 467 (*λ-system and intersection*),
Lemma 470 (*λ-system with intersection*).

The proof of Lemma 472 (*λ-system generated by π-system*)
cites explicitly:
Lemma 417 (*closedness under finite operations*),

Definition 428 (*π-system*),
Lemma 464 (*generated λ-system is minimum*),
Lemma 471 (*λ-system is closed under intersection*).

The proof of Lemma 475 (*equivalent definition of σ-algebra*)
cites explicitly:
Lemma 408 (*nonempty and with empty or full*),
Lemma 409 (*with empty and full*),
Lemma 423 (*closedness under countable union and intersection*),
Definition 474 (*σ-algebra*).

The proof of Lemma 477 (*σ-algebra is set algebra*)
cites explicitly:
Definition 437 (*set algebra*),
Definition 474 (*σ-algebra*).

The proof of Lemma 478 (*σ-algebra is closed under set difference*)
cites explicitly:
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 440 (*set algebra is closed under local complement*),
Lemma 477 (*σ-algebra is set algebra*).

The proof of Lemma 479 (*other properties of σ-algebra*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*).

The proof of Lemma 480 (*partition of countable union in σ-algebra*)
cites explicitly:
Lemma 215 (*partition of countable union*),
Lemma 446 (*partition of countable union in set algebra*),
Definition 474 (*σ-algebra*),
Lemma 477 (*σ-algebra is set algebra*).

The proof of Lemma 481 (*intersection of σ-algebras*)
cites explicitly:
Definition 474 (*σ-algebra*).

The proof of Lemma 483 (*generated σ-algebra is minimum*)
cites explicitly:
Lemma 481 (*intersection of σ-algebras*),
Definition 482 (*generated σ-algebra*).

The proof of Lemma 484 (*σ-algebra generation is monotone*)
cites explicitly:
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 485 (*σ-algebra generation is idempotent*)
cites explicitly:
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 486 (*σ-algebra is π-system*)
cites explicitly:
Definition 428 (*π-system*),
Lemma 475 (*equivalent definition of σ-algebra*).

The proof of Lemma 487 (*σ-algebra contains π-system*)
cites explicitly:
Lemma 433 (*generated π-system is minimum*),

Lemma 483 (*generated σ-algebra is minimum*),
Lemma 486 (*σ-algebra is π-system*).

The proof of Lemma 488 (*π-system contains σ-algebra*)
cites explicitly:
Lemma 408 (*nonempty and with empty or full*),
Lemma 426 (*closedness under countable disjoint union and countable union*),
Definition 428 (*π-system*),
Lemma 433 (*generated π-system is minimum*),
Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 489 (*σ-algebra generated by π-system*)
cites explicitly:
Lemma 433 (*generated π-system is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 487 (*σ-algebra contains π-system*).

The proof of Lemma 490 (*σ-algebra contains set algebra*)
cites explicitly:
Lemma 443 (*generated set algebra is minimum*),
Lemma 477 (*σ-algebra is set algebra*),
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 491 (*set algebra contains σ-algebra*)
cites explicitly:
Lemma 427 (*closedness under countable monotone union and countable union*),
Definition 437 (*set algebra*),
Lemma 443 (*generated set algebra is minimum*),
Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 492 (*σ-algebra generated by set algebra*)
cites explicitly:
Lemma 443 (*generated set algebra is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 490 (*σ-algebra contains set algebra*).

The proof of Lemma 493 (*σ-algebra is monotone class*)
cites explicitly:
Definition 448 (*monotone class*),
Lemma 479 (*other properties of σ-algebra*).

The proof of Lemma 494 (*σ-algebra contains monotone class*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 493 (*σ-algebra is monotone class*).

The proof of Lemma 495 (*monotone class contains σ-algebra*)
cites explicitly:
Lemma 427 (*closedness under countable monotone union and countable union*),
Definition 448 (*monotone class*),
Lemma 451 (*generated monotone class is minimum*),

Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 496 (*σ-algebra generated by monotone class*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 494 (*σ-algebra contains monotone class*).

The proof of Lemma 497 (*σ-algebra is λ-system*)
cites explicitly:
Definition 459 (*λ-system*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 479 (*other properties of σ-algebra*).

The proof of Lemma 498 (*σ-algebra contains λ-system*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 497 (*σ-algebra is λ-system*).

The proof of Lemma 499 (*λ-system contains σ-algebra*)
cites explicitly:
Lemma 426 (*closedness under countable disjoint union and countable union*),
Definition 459 (*λ-system*),
Lemma 464 (*generated λ-system is minimum*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*).

The proof of Lemma 500 (*σ-algebra generated by λ-system*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 498 (*σ-algebra contains λ-system*).

The proof of Lemma 501 (*other σ-algebra generator*)
cites explicitly:
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 485 (*σ-algebra generation is idempotent*).

The proof of Lemma 502 (*complete generated σ-algebra*)
cites explicitly:
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 501 (*other σ-algebra generator*).

The proof of Lemma 503 (*countable σ-algebra generator*)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 501 (*other σ-algebra generator*).

The proof of Lemma 505 (*set algebra generated by product of σ-algebras*)
cites explicitly:
Definition 217 (*product of subsets of parties*),

Lemma 447 (*explicit set algebra*),
Lemma 475 (*equivalent definition of σ-algebra*).

The proof of Lemma 506 (*π-system and λ-system is σ-algebra*)
cites explicitly:
Definition 428 (*π-system*),
Lemma 466 (*λ-system generation is idempotent*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 498 (*σ-algebra contains λ-system*),
Lemma 499 (*λ-system contains σ-algebra*).

The proof of Theorem 508 (*Dynkin π–λ theorem*)
cites explicitly:
Lemma 464 (*generated λ-system is minimum*),
Lemma 472 (*λ-system generated by π-system*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 500 (*σ-algebra generated by λ-system*),
Lemma 506 (*π-system and λ-system is σ-algebra*).

The proof of Lemma 510 (*usage of Dynkin π–λ theorem*)
cites explicitly:
Lemma 433 (*generated π-system is minimum*),
Lemma 465 (*λ-system generation is monotone*),
Lemma 466 (*λ-system generation is idempotent*),
Definition 482 (*generated σ-algebra*),
Lemma 484 (*σ-algebra generation is monotone*),
Theorem 508 (*Dynkin π–λ theorem*).

The proof of Lemma 511 (*algebra and monotone class is σ-algebra*)
cites explicitly:
Definition 437 (*set algebra*),
Lemma 453 (*monotone class generation is idempotent*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 494 (*σ-algebra contains monotone class*),
Lemma 495 (*monotone class contains σ-algebra*).

The proof of Theorem 513 (*monotone class*)
cites explicitly:
Lemma 451 (*generated monotone class is minimum*),
Lemma 458 (*monotone class generated by set algebra*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 496 (*σ-algebra generated by monotone class*),
Lemma 511 (*algebra and monotone class is σ-algebra*).

The proof of Lemma 515 (*usage of monotone class theorem*)
cites explicitly:
Lemma 443 (*generated set algebra is minimum*),
Lemma 452 (*monotone class generation is monotone*),
Lemma 453 (*monotone class generation is idempotent*),
Definition 482 (*generated σ-algebra*),
Lemma 484 (*σ-algebra generation is monotone*),
Theorem 513 (*monotone class*).

The proof of Lemma 518 (*some Borel subsets*)
cites explicitly:
Definition 249 (*topological space*),

Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*),
Definition 517 (*Borel σ-algebra*).

The proof of Lemma 519 (*countable Borel σ-algebra generator*)
cites explicitly:
Lemma 503 (*countable σ-algebra generator*),
Definition 517 (*Borel σ-algebra*).

The proof of Lemma 523 (*inverse σ-algebra*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 522 (*measurable function*).

The proof of Lemma 524 (*image σ-algebra*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 522 (*measurable function*).

The proof of Lemma 525 (*identity function is measurable*)
cites explicitly:
Definition 522 (*measurable function*).

The proof of Lemma 526 (*constant function is measurable*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*).

The proof of Lemma 527 (*inverse image of generating family*)
cites explicitly:
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 523 (*inverse σ-algebra*),
Lemma 524 (*image σ-algebra*).

The proof of Lemma 528 (*equivalent definition of measurable function*)
cites explicitly:
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 485 (*σ-algebra generation is idempotent*),
Definition 522 (*measurable function*),
Lemma 527 (*inverse image of generating family*).

The proof of Lemma 529 (*continuous is measurable*)
cites explicitly:
Lemma 483 (*generated σ-algebra is minimum*),
Definition 517 (*Borel σ-algebra*),
Definition 522 (*measurable function*),
Lemma 528 (*equivalent definition of measurable function*).

The proof of Lemma 530 (*compatibility of measurability with composition*)
cites explicitly:
Definition 522 (*measurable function*).

The proof of Lemma 532 (*trace σ-algebra*)
cites explicitly:

Definition 216 (*trace of subsets of parties*),
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 522 (*measurable function*).

The proof of Lemma 533 (*measurability of measurable subspace*)
cites explicitly:
Definition 216 (*trace of subsets of parties*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 532 (*trace σ-algebra*).

The proof of Lemma 534 (*generating measurable subspace*)
cites explicitly:
Definition 216 (*trace of subsets of parties*),
Lemma 527 (*inverse image of generating family*).

The proof of Lemma 535 (*Borel sub-σ-algebra*)
cites explicitly:
Definition 517 (*Borel σ-algebra*),
Lemma 533 (*measurability of measurable subspace*),
Lemma 534 (*generating measurable subspace*).

The proof of Lemma 536 (*characterization of Borel subsets*)
cites explicitly:
Lemma 209 (*compatibility of pseudopartition with intersection*),
Definition 474 (*σ-algebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 517 (*Borel σ-algebra*),
Lemma 535 (*Borel sub-σ-algebra*).

The proof of Lemma 537 (*source restriction of measurable function*)
cites explicitly:
Definition 216 (*trace of subsets of parties*),
Definition 522 (*measurable function*),
Lemma 530 (*compatibility of measurability with composition*),
Lemma 532 (*trace σ-algebra*).

The proof of Lemma 538 (*destination restriction of measurable function*)
cites explicitly:
Definition 522 (*measurable function*).

The proof of Lemma 539 (*measurability of function defined on a pseudopartition*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 522 (*measurable function*).

The proof of Lemma 542 (*product of measurable subsets is measurable*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Lemma 483 (*generated σ-algebra is minimum*),
Definition 541 (*tensor product of σ-algebras*).

The proof of Lemma 543 (*measurability of function to product space*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 522 (*measurable function*),

Lemma 528 (*equivalent definition of measurable function*),
Definition 541 (*tensor product of σ-algebras*),
Lemma 542 (*product of measurable subsets is measurable*).

The proof of Lemma 544 (*canonical projection is measurable*)
cites explicitly:
Definition 522 (*measurable function*),
Lemma 543 (*measurability of function to product space*).

The proof of Lemma 545 (*permutation is measurable*)
cites explicitly:
Lemma 543 (*measurability of function to product space*),
Lemma 544 (*canonical projection is measurable*).

The proof of Lemma 546 (*generating product measurable space*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 484 (*σ-algebra generation is monotone*),
Definition 522 (*measurable function*),
Lemma 527 (*inverse image of generating family*),
Definition 541 (*tensor product of σ-algebras*),
Lemma 544 (*canonical projection is measurable*).

The proof of Lemma 549 (*section of product*)
cites explicitly:
Definition 548 (*section in Cartesian product*).

The proof of Lemma 550 (*compatibility of section with set operations*)
cites explicitly:
Definition 548 (*section in Cartesian product*).

The proof of Lemma 551 (*measurability of section*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Definition 474 (*σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*),
Definition 516 (*measurable space*),
Definition 541 (*tensor product of σ-algebras*),
Lemma 549 (*section of product*),
Lemma 550 (*compatibility of section with set operations*).

The proof of Lemma 552 (*countable union of sections is measurable*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Lemma 550 (*compatibility of section with set operations*),
Lemma 551 (*measurability of section*).

The proof of Lemma 553 (*countable intersection of sections is measurable*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 550 (*compatibility of section with set operations*),
Lemma 551 (*measurability of section*).

The proof of Lemma 554 (*indicator of section*)
cites explicitly:
Definition 548 (*section in Cartesian product*).

The proof of Lemma 555 (*measurability of function from product space*)
cites explicitly:
Definition 522 (*measurable function*),
Lemma 551 (*measurability of section*).

The proof of Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$)
cites explicitly:
Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$),
Definition 474 (*σ-algebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 482 (*generated σ-algebra*),
Lemma 501 (*other σ-algebra generator*),
Lemma 518 (*some Borel subsets*),
Lemma 519 (*countable Borel σ-algebra generator*).

The proof of Lemma 559 (*countable generator of Borel σ-algebra of* $\mathbb{R}$)
cites explicitly:
Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$),
Theorem 359 ($\mathbb{R}$ *is second-countable*),
Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$).

The proof of Lemma 560 (*Borel σ-algebra of* $\overline{\mathbb{R}}$)
cites explicitly:
Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 482 (*generated σ-algebra*),
Lemma 501 (*other σ-algebra generator*),
Lemma 519 (*countable Borel σ-algebra generator*).

The proof of Lemma 561 (*Borel subsets of* $\overline{\mathbb{R}}$ *and* $\mathbb{R}$)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 517 (*Borel σ-algebra*),
Lemma 535 (*Borel sub-σ-algebra*),
Lemma 536 (*characterization of Borel subsets*).

The proof of Lemma 563 (*Borel σ-algebra of* $\mathbb{R}_+$)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 535 (*Borel sub-σ-algebra*),
Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$).

The proof of Lemma 564 (*Borel σ-algebra of* $\overline{\mathbb{R}}_+$)
cites explicitly:
Lemma 534 (*generating measurable subspace*),
Lemma 560 (*Borel σ-algebra of* $\overline{\mathbb{R}}$).

The proof of Lemma 565 (*Borel σ-algebra of* $\mathbb{R}^n$)
cites explicitly:
Lemma 266 (*complete countable topological basis of product space*),
Lemma 360 ($\mathbb{R}^n$ *is second-countable*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 502 (*complete generated σ-algebra*),
Lemma 519 (*countable Borel σ-algebra generator*),
Lemma 546 (*generating product measurable space*),
Lemma 559 (*countable generator of Borel σ-algebra of* $\mathbb{R}$).

The proof of Lemma 569 (*measurability of indicator function*)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 516 (*measurable space*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*).

The proof of Lemma 570 (*measurability of numeric function to $\mathbb{R}$*)
cites explicitly:
Definition 517 (*Borel σ-algebra*),
Lemma 518 (*some Borel subsets*),
Definition 522 (*measurable function*),
Lemma 528 (*equivalent definition of measurable function*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*).

The proof of Lemma 571 (*inverse image is measurable in $\mathbb{R}$*)
cites explicitly:
Lemma 518 (*some Borel subsets*),
Definition 522 (*measurable function*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*).

The proof of Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*)
cites explicitly:
Lemma 228 (*$\mathbb{K}$ is $\mathbb{K}$-algebra*),
Lemma 231 (*algebra of functions to algebra*),
Lemma 236 (*closed under algebra operations is subalgebra*),
Lemma 526 (*constant function is measurable*),
Lemma 529 (*continuous is measurable*),
Lemma 530 (*compatibility of measurability with composition*),
Lemma 543 (*measurability of function to product space*),
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*).

The proof of Lemma 574 (*$\mathcal{M}_{\mathbb{R}}$ is vector space*)
cites explicitly:
Definition 226 (*algebra over a field*),
Definition 233 (*subalgebra*),
Lemma 236 (*closed under algebra operations is subalgebra*),
Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*).

The proof of Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*)
cites explicitly:
Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*),
Definition 522 (*measurable function*),
Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*).

The proof of Lemma 578 (*measurability of numeric function*)
cites explicitly:
Definition 517 (*Borel σ-algebra*),
Lemma 518 (*some Borel subsets*),
Definition 522 (*measurable function*),
Lemma 528 (*equivalent definition of measurable function*),

Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*).

The proof of Lemma 579 (*inverse image is measurable*)
cites explicitly:
Lemma 518 (*some Borel subsets*),
Definition 522 (*measurable function*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*).

The proof of Lemma 580 (*$\mathcal{M}$ is closed under finite part*)
cites explicitly:
Definition 207 (*pseudopartition*),
Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*),
Definition 397 (*finite part*),
Lemma 398 (*finite part is finite*),
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Lemma 526 (*constant function is measurable*),
Lemma 539 (*measurability of function defined on a pseudopartition*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Lemma 579 (*inverse image is measurable*).

The proof of Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*)
cites explicitly:
Definition 207 (*pseudopartition*),
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Definition 397 (*finite part*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 517 (*Borel σ-algebra*),
Definition 522 (*measurable function*),
Lemma 526 (*constant function is measurable*),
Lemma 539 (*measurability of function defined on a pseudopartition*),
Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Lemma 579 (*inverse image is measurable*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*).

The proof of Lemma 582 (*$\mathcal{M}$ is closed under finite sum when defined*)
cites explicitly:
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*).

The proof of Lemma 583 (*$\mathcal{M}$ is closed under multiplication*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*),
Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*),
Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Definition 397 (*finite part*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 517 (*Borel σ-algebra*),
Definition 522 (*measurable function*),
Lemma 526 (*constant function is measurable*),
Lemma 539 (*measurability of function defined on a pseudopartition*),
Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Lemma 578 (*measurability of numeric function*),

Lemma 579 (*inverse image is measurable*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*).

The proof of Lemma 584 (*$\mathcal{M}$ is closed under finite product*)
cites explicitly:
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

The proof of Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*)
cites explicitly:
Lemma 526 (*constant function is measurable*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

The proof of Lemma 586 (*$\mathcal{M}$ is closed under infimum*)
cites explicitly:
Statement(s) from [17],
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 578 (*measurability of numeric function*).

The proof of Lemma 587 (*$\mathcal{M}$ is closed under supremum*)
cites explicitly:
Statement(s) from [17],
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 578 (*measurability of numeric function*).

The proof of Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*)
cites explicitly:
Lemma 378 (*limit inferior*),
Lemma 586 (*$\mathcal{M}$ is closed under infimum*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*).

The proof of Lemma 589 (*$\mathcal{M}$ is closed under limit superior*)
cites explicitly:
Lemma 383 (*limit superior*),
Lemma 586 (*$\mathcal{M}$ is closed under infimum*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*).

The proof of Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*)
cites explicitly:
Lemma 396 (*limit inferior, limit superior and pointwise convergence*),
Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*),
Lemma 589 (*$\mathcal{M}$ is closed under limit superior*).

The proof of Lemma 591 (*measurability and masking*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*),
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Lemma 569 (*measurability of indicator function*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

The proof of Lemma 592 (*measurability of restriction*)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 516 (*measurable space*),
Definition 522 (*measurable function*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*).

The proof of Lemma 594 (*measurability of nonnegative and nonpositive parts*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*),
Lemma 401 (*nonnegative and nonpositive parts are nonnegative*),
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 526 (*constant function is measurable*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 595 (*$\mathcal{M}_+$ is closed under finite part*)
cites explicitly:
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 596 (*$\mathcal{M}$ is closed under absolute value*)
cites explicitly:
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Lemma 317 (*absolute value in $\overline{\mathbb{R}}$ is continuous*),
Lemma 529 (*continuous is measurable*),
Lemma 530 (*compatibility of measurability with composition*),
Definition 567 (*$\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 597 (*$\mathcal{M}_+$ is closed under addition*)
cites explicitly:
Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*)
cites explicitly:
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*)
cites explicitly:
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 600 (*$\mathcal{M}_+$ is closed under infimum*)
cites explicitly:
Lemma 376 (*infimum of bounded sequence is bounded*),
Lemma 586 (*$\mathcal{M}$ is closed under infimum*).

The proof of Lemma 601 (*$\mathcal{M}_+$ is closed under supremum*)
cites explicitly:
Lemma 377 (*supremum of bounded sequence is bounded*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*).

The proof of Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*)
cites explicitly:

Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*).

The proof of Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*)
cites explicitly:
Lemma 322 (*series are convergent in $\overline{\mathbb{R}}_+$*),
Lemma 597 (*$\mathcal{M}_+$ is closed under addition*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*).

The proof of Lemma 605 (*measurability of tensor product of numeric functions*)
cites explicitly:
Lemma 530 (*compatibility of measurability with composition*),
Lemma 544 (*canonical projection is measurable*),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Lemma 584 (*$\mathcal{M}$ is closed under finite product*),
Definition 604 (*tensor product of numeric functions*).

The proof of Lemma 610 (*σ-additivity implies additivity*)
cites explicitly:
Definition 607 (*additivity over measurable space*),
Definition 608 (*σ-additivity over measurable space*).

The proof of Lemma 613 (*measure over countable pseudopartition*)
cites explicitly:
Lemma 209 (*compatibility of pseudopartition with intersection*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*).

The proof of Lemma 614 (*measure is monotone*)
cites explicitly:
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Lemma 478 (*σ-algebra is closed under set difference*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*).

The proof of Lemma 615 (*measure satisfies the finite Boole inequality*)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 478 (*σ-algebra is closed under set difference*),
Definition 516 (*measurable space*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 614 (*measure is monotone*).

The proof of Lemma 617 (*measure is continuous from below*)
cites explicitly:
Lemma 480 (*partition of countable union in σ-algebra*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 614 (*measure is monotone*),
Definition 616 (*continuity from below*).

The proof of Lemma 619 (*measure is continuous from above*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 478 (*σ-algebra is closed under set difference*),

Lemma 614 (*measure is monotone*),
Definition 616 (*continuity from below*),
Lemma 617 (*measure is continuous from below*),
Definition 618 (*continuity from above*).

The proof of Lemma 620 (*measure satisfies the Boole inequality*)
cites explicitly:
Statement(s) from [17],
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 617 (*measure is continuous from below*).

The proof of Lemma 621 (*equivalent definition of measure*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 607 (*additivity over measurable space*),
Definition 608 (*σ-additivity over measurable space*),
Lemma 610 (*σ-additivity implies additivity*),
Definition 611 (*measure*),
Definition 616 (*continuity from below*),
Lemma 617 (*measure is continuous from below*).

The proof of Lemma 623 (*finite measure is bounded*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 614 (*measure is monotone*).

The proof of Lemma 625 (*equivalent definition of σ-finite measure*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Definition 624 (*σ-finite measure*).

The proof of Lemma 627 (*finite measure is σ-finite*)
cites explicitly:
Definition 622 (*finite measure*),
Definition 624 (*σ-finite measure*).

The proof of Lemma 628 (*trace measure*)
cites explicitly:
Lemma 532 (*trace σ-algebra*),
Lemma 533 (*measurability of measurable subspace*),
Definition 611 (*measure*).

The proof of Lemma 629 (*restricted measure*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 611 (*measure*).

The proof of Lemma 634 (*equivalent definition of considerable subset*)
cites explicitly:
Definition 611 (*measure*),

Definition 631 (*negligible subset*),
Definition 633 (*considerable subset*).

The proof of Lemma 636 (*negligibility of measurable subset*)
cites explicitly:
Definition 631 (*negligible subset*).

The proof of Lemma 637 (*empty set is negligible*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Lemma 636 (*negligibility of measurable subset*).

The proof of Lemma 638 (*compatibility of null measure with countable union*)
cites explicitly:
Statement(s) from [17],
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 620 (*measure satisfies the Boole inequality*).

The proof of Lemma 639 ($\mathbf{N}$ *is closed under countable union*)
cites explicitly:
Definition 631 (*negligible subset*),
Lemma 638 (*compatibility of null measure with countable union*).

The proof of Lemma 640 (*subset of negligible is negligible*)
cites explicitly:
Definition 631 (*negligible subset*).

The proof of Lemma 643 (*everywhere implies almost everywhere*)
cites explicitly:
Lemma 637 (*empty set is negligible*),
Definition 641 (*property almost satisfied*).

The proof of Lemma 644 (*everywhere implies almost everywhere for almost the same*)

cites explicitly:
Lemma 640 (*subset of negligible is negligible*),
Definition 641 (*property almost satisfied*).

The proof of Lemma 646 (*extended almost modus ponens*)
cites explicitly:
Lemma 639 ($\mathbf{N}$ *is closed under countable union*),
Lemma 640 (*subset of negligible is negligible*),
Definition 641 (*property almost satisfied*).

The proof of Lemma 647 (*almost modus ponens*)
cites explicitly:
Lemma 643 (*everywhere implies almost everywhere*),
Lemma 646 (*extended almost modus ponens*).

The proof of Lemma 651 (*compatibility of almost binary relation with reflexivity*)
cites explicitly:
Definition 641 (*property almost satisfied*),

Lemma 643 (*everywhere implies almost everywhere*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Definition 649 (*almost definition*),
Definition 650 (*almost binary relation*).

The proof of Lemma 652 (*compatibility of almost binary relation with symmetry*)
cites explicitly:
Definition 641 (*property almost satisfied*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Lemma 647 (*almost modus ponens*),
Definition 649 (*almost definition*),
Definition 650 (*almost binary relation*).

The proof of Lemma 653 (*compatibility of almost binary relation with antisymmetry*)

cites explicitly:
Lemma 639 (**N** *is closed under countable union*),
Lemma 640 (*subset of negligible is negligible*),
Definition 641 (*property almost satisfied*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Definition 649 (*almost definition*),
Definition 650 (*almost binary relation*).

The proof of Lemma 654 (*compatibility of almost binary relation with transitivity*)
cites explicitly:
Lemma 639 (**N** *is closed under countable union*),
Lemma 640 (*subset of negligible is negligible*),
Definition 641 (*property almost satisfied*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Definition 649 (*almost definition*),
Definition 650 (*almost binary relation*).

The proof of Lemma 656 (*almost equivalence is equivalence relation*)
cites explicitly:
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*).

The proof of Lemma 657 (*almost equality is equivalence relation*)
cites explicitly:
Lemma 656 (*almost equivalence is equivalence relation*).

The proof of Lemma 658 (*almost order is order relation*)
cites explicitly:
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*).

The proof of Lemma 659 (*compatibility of almost binary relation with operator*)
cites explicitly:
Lemma 639 (**N** *is closed under countable union*),
Definition 641 (*property almost satisfied*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Definition 649 (*almost definition*),
Definition 650 (*almost binary relation*).

The proof of Lemma 660 (*compatibility of almost equality with operator*)
cites explicitly:
Lemma 659 (*compatibility of almost binary relation with operator*).

The proof of Lemma 661 (*compatibility of almost inequality with operator*)
cites explicitly:
Lemma 659 (*compatibility of almost binary relation with operator*).

The proof of Lemma 664 (*definiteness implies almost definiteness*)
cites explicitly:
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Lemma 647 (*almost modus ponens*).

The proof of Lemma 668 (*uniqueness of measures extended from a π-system*)
cites explicitly:
Definition 207 (*pseudopartition*),
Definition 428 (*π-system*),
Lemma 435 (*π-system generation is idempotent*),
Lemma 460 (*equivalent definition of λ-system*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 510 (*usage of Dynkin π–λ theorem*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 614 (*measure is monotone*),
Definition 616 (*continuity from below*),
Lemma 617 (*measure is continuous from below*).

The proof of Lemma 669 (*trivial measure*)
cites explicitly:
Definition 611 (*measure*).

The proof of Lemma 670 (*equivalent definition of trivial measure*)
cites explicitly:
Definition 611 (*measure*),
Lemma 614 (*measure is monotone*),
Lemma 669 (*trivial measure*).

The proof of Lemma 671 (*counting measure*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*).

The proof of Lemma 672 (*finiteness of counting measure*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 622 (*finite measure*),
Lemma 671 (*counting measure*).

The proof of Lemma 673 (*σ-finite counting measure*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Definition 624 (*σ-finite measure*),

Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 671 (*counting measure*).

The proof of Lemma 676 (*equivalent definition of Dirac measure*)
cites explicitly:
Lemma 671 (*counting measure*),
Definition 675 (*Dirac measure*).

The proof of Lemma 677 (*Dirac measure is finite*)
cites explicitly:
Lemma 671 (*counting measure*),
Lemma 672 (*finiteness of counting measure*),
Definition 675 (*Dirac measure*).

The proof of Lemma 679 (*summability on summability domain*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Definition 678 (*summability domain*).

The proof of Lemma 680 (*measurability of summability domain*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 579 (*inverse image is measurable*).

The proof of Lemma 681 (*negligibility of summability domain*)
cites explicitly:
Definition 641 (*property almost satisfied*),
Lemma 679 (*summability on summability domain*).

The proof of Lemma 682 (*almost sum*)
cites explicitly:
Definition 282 (*addition in* $\overline{\mathbb{R}}$),
Lemma 569 (*measurability of indicator function*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Definition 641 (*property almost satisfied*),
Lemma 680 (*measurability of summability domain*),
Lemma 681 (*negligibility of summability domain*).

The proof of Lemma 683 (*compatibility of almost sum with almost equality*)
cites explicitly:
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 637 (*empty set is negligible*),
Lemma 657 (*almost equality is equivalence relation*),
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 681 (*negligibility of summability domain*),
Lemma 682 (*almost sum*).

The proof of Lemma 685 (*almost sum is sum*)
cites explicitly:
Definition 678 (*summability domain*),
Lemma 682 (*almost sum*).

The proof of Lemma 686 (*absolute value is almost definite*)
cites explicitly:
Lemma 304 (*absolute value in* $\overline{\mathbb{R}}$ *is definite*),
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 664 (*definiteness implies almost definiteness*).

The proof of Lemma 687 (*masking almost nowhere*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Definition 631 (*negligible subset*),
Definition 641 (*property almost satisfied*).

The proof of Lemma 688 (*finite nonnegative part*)
cites explicitly:
Definition 397 (*finite part*),
Lemma 400 (*equivalent definition of nonnegative and nonpositive parts*),
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Lemma 591 (*measurability and masking*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 595 (*$\mathcal{M}_+$ is closed under finite part*),
Definition 611 (*measure*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 636 (*negligibility of measurable subset*),
Definition 641 (*property almost satisfied*),
Lemma 687 (*masking almost nowhere*).

The proof of Lemma 692 (*length is nonnegative*)
cites explicitly:
Definition 691 (*length of interval*).

The proof of Lemma 693 (*length is homogeneous*)
cites explicitly:
Definition 691 (*length of interval*).

The proof of Lemma 694 (*length of partition*)
cites explicitly:
Definition 691 (*length of interval*),
Lemma 693 (*length is homogeneous*).

The proof of Lemma 696 (*set of countable cover with bounded open intervals is nonempty*)
cites explicitly:
Definition 695 (*set of countable cover with bounded open intervals*).

The proof of Lemma 699 (*$\lambda^\star$ is nonnegative*)
cites explicitly:
Lemma 692 (*length is nonnegative*).

The proof of Lemma 700 (*$\lambda^\star$ is homogeneous*)
cites explicitly:
Statement(s) from [17],
Definition 691 (*length of interval*),
Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*),
Lemma 699 (*$\lambda^\star$ is nonnegative*).

The proof of Lemma 701 (*$\lambda^\star$ is monotone*)
cites explicitly:
Statement(s) from [17],
Definition 695 (*set of countable cover with bounded open intervals*),
Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*).

The proof of Lemma 702 ($\lambda^\star$ *is* σ*-subadditive*)
cites explicitly:
Statement(s) from [17],
Lemma 212 (*definition of double countable union*),
Lemma 213 (*double countable union*),
Lemma 325 (*definition of double series in* $\overline{\mathbb{R}}_+$),
Lemma 326 (*double series in* $\overline{\mathbb{R}}_+$),
Definition 695 (*set of countable cover with bounded open intervals*),
Definition 697 ($\lambda^\star$*, Lebesgue measure candidate*).

The proof of Lemma 703 ($\lambda^\star$ *generalizes length of interval*)
cites explicitly:
Statement(s) from [17],
Lemma 272 (*finite cover of compact interval*),
Definition 691 (*length of interval*),
Definition 695 (*set of countable cover with bounded open intervals*),
Definition 697 ($\lambda^\star$*, Lebesgue measure candidate*),
Lemma 700 ($\lambda^\star$ *is homogeneous*),
Lemma 701 ($\lambda^\star$ *is monotone*).

The proof of Lemma 707 (*equivalent definition of* $\mathcal{L}$)
cites explicitly:
Lemma 209 (*compatibility of pseudopartition with intersection*),
Lemma 279 (*order in* $\overline{\mathbb{R}}$ *is total*),
Lemma 702 ($\lambda^\star$ *is* σ*-subadditive*),
Definition 705 ($\mathcal{L}$*, Lebesgue* σ*-algebra*).

The proof of Lemma 708 ($\mathcal{L}$ *is closed under complement*)
cites explicitly:
Lemma 320 (*addition in* $\overline{\mathbb{R}}_+$ *is commutative*),
Definition 705 ($\mathcal{L}$*, Lebesgue* σ*-algebra*).

The proof of Lemma 709 ($\mathcal{L}$ *is closed under finite union*)
cites explicitly:
Lemma 702 ($\lambda^\star$ *is* σ*-subadditive*),
Definition 705 ($\mathcal{L}$*, Lebesgue* σ*-algebra*),
Lemma 707 (*equivalent definition of* $\mathcal{L}$).

The proof of Lemma 710 ($\mathcal{L}$ *is closed under finite intersection*)
cites explicitly:
Lemma 708 ($\mathcal{L}$ *is closed under complement*),
Lemma 709 ($\mathcal{L}$ *is closed under finite union*).

The proof of Lemma 711 ($\mathcal{L}$ *is set algebra*)
cites explicitly:
Lemma 414 (*closedness under intersection and set difference*),
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 700 ($\lambda^\star$ *is homogeneous*),
Lemma 708 ($\mathcal{L}$ *is closed under complement*),
Lemma 710 ($\mathcal{L}$ *is closed under finite intersection*).

The proof of Lemma 712 ($\lambda^\star$ *is additive on* $\mathcal{L}$)
cites explicitly:
Lemma 209 (*compatibility of pseudopartition with intersection*),
Definition 705 ($\mathcal{L}$*, Lebesgue* σ*-algebra*).

The proof of Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*)
cites explicitly:
Definition 608 (*σ-additivity over measurable space*),
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*).

The proof of Lemma 715 (*partition of countable union in $\mathcal{L}$*)
cites explicitly:
Lemma 446 (*partition of countable union in set algebra*),
Lemma 711 (*$\mathcal{L}$ is set algebra*).

The proof of Lemma 716 (*$\mathcal{L}$ is closed under countable union*)
cites explicitly:
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*),
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 709 (*$\mathcal{L}$ is closed under finite union*),
Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*),
Lemma 715 (*partition of countable union in $\mathcal{L}$*).

The proof of Lemma 717 (*rays are Lebesgue-measurable*)
cites explicitly:
Statement(s) from [17],
Lemma 246 (*intervals are closed under finite intersection*),
Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*),
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 708 (*$\mathcal{L}$ is closed under complement*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

The proof of Lemma 718 (*intervals are Lebesgue-measurable*)
cites explicitly:
Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*),
Lemma 717 (*rays are Lebesgue-measurable*).

The proof of Lemma 719 (*$\mathcal{L}$ is σ-algebra*)
cites explicitly:
Definition 437 (*set algebra*),
Definition 474 (*σ-algebra*),
Lemma 711 (*$\mathcal{L}$ is set algebra*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

The proof of Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*)
cites explicitly:
Definition 611 (*measure*),
Lemma 699 (*$\lambda^\star$ is nonnegative*),
Lemma 700 (*$\lambda^\star$ is homogeneous*),
Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*),
Lemma 719 (*$\mathcal{L}$ is σ-algebra*).

The proof of Lemma 721 (*$\mathcal{B}(\mathbb{R})$ is sub-σ-algebra of $\mathcal{L}$*)
cites explicitly:

Lemma 484 (*σ-algebra generation is monotone*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 718 (*intervals are Lebesgue-measurable*).

The proof of Lemma 722 (*$\lambda^\star$ is measure on $\mathcal{B}(\mathbb{R})$*)
cites explicitly:
Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*),
Lemma 721 (*$\mathcal{B}(\mathbb{R})$ is sub-σ-algebra of $\mathcal{L}$*).

The proof of Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*)
cites explicitly:
Lemma 246 (*intervals are closed under finite intersection*),
Definition 428 (*π-system*),
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 619 (*measure is continuous from above*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 722 (*$\lambda^\star$ is measure on $\mathcal{B}(\mathbb{R})$*).

The proof of Lemma 726 (*Lebesgue measure generalizes length of interval*)
cites explicitly:
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

The proof of Lemma 728 (*Lebesgue measure is σ-finite*)
cites explicitly:
Definition 624 (*σ-finite measure*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

The proof of Lemma 729 (*Lebesgue measure is diffuse*)
cites explicitly:
Definition 626 (*diffuse measure*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

The proof of Lemma 733 (*indicator and support are each other inverse*)
cites explicitly:
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 734 (*$\mathcal{IF}$ is measurable*)
cites explicitly:
Lemma 569 (*measurability of indicator function*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_\mathbb{R}$*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 735 (*$\mathcal{IF}$ is σ-additive*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*)
cites explicitly:
Lemma 218 (*restriction is masking*),
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 739 (*$\mathcal{IF}$ is closed under restriction*)
cites explicitly:
Lemma 533 (*measurability of measurable subspace*),
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*).

The proof of Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*)
cites explicitly:
Lemma 733 (*indicator and support are each other inverse*),
Definition 740 (*integral in $\mathcal{IF}$*).

The proof of Lemma 742 (*integral in $\mathcal{IF}$ is additive*)
cites explicitly:
Definition 607 (*additivity over measurable space*),
Lemma 621 (*equivalent definition of measure*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*).

The proof of Lemma 743 (*integral in $\mathcal{IF}$ over subset*)
cites explicitly:
Lemma 218 (*restriction is masking*),
Lemma 628 (*trace measure*),
Lemma 733 (*indicator and support are each other inverse*),
Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*),
Lemma 739 (*$\mathcal{IF}$ is closed under restriction*),
Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*).

The proof of Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*)
cites explicitly:
Definition 474 (*σ-algebra*),
Definition 516 (*measurable space*),
Definition 611 (*measure*),
Lemma 733 (*indicator and support are each other inverse*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*),
Lemma 742 (*integral in $\mathcal{IF}$ is additive*),
Lemma 743 (*integral in $\mathcal{IF}$ over subset*).

The proof of Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*)
cites explicitly:
Lemma 671 (*counting measure*),
Lemma 733 (*indicator and support are each other inverse*),
Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*).

The proof of Lemma 749 (*$\mathcal{SF}$ simple representation*)
cites explicitly:
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*),
Lemma 733 (*indicator and support are each other inverse*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*).

The proof of Lemma 752 (*$\mathcal{SF}$ canonical representation*)
cites explicitly:

Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 749 (*$\mathcal{SF}$ simple representation*).

The proof of Lemma 754 (*$\mathcal{SF}$ disjoint representation*)
cites explicitly:
Lemma 749 (*$\mathcal{SF}$ simple representation*),
Lemma 752 (*$\mathcal{SF}$ canonical representation*).

The proof of Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*)
cites explicitly:
Lemma 752 (*$\mathcal{SF}$ canonical representation*),
Lemma 754 (*$\mathcal{SF}$ disjoint representation*).

The proof of Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*)
cites explicitly:
Statement(s) from [17],
Lemma 235 (*vector subspace and closed under multiplication is subalgebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Lemma 749 (*$\mathcal{SF}$ simple representation*),
Lemma 754 (*$\mathcal{SF}$ disjoint representation*).

The proof of Lemma 759 (*$\mathcal{SF}$ is measurable*)
cites explicitly:
Lemma 569 (*measurability of indicator function*),
Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*).

The proof of Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*)
cites explicitly:
Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*).

The proof of Lemma 761 (*$\mathcal{SF}$ is closed under restriction*)
cites explicitly:
Lemma 739 (*$\mathcal{IF}$ is closed under restriction*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*).

The proof of Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*)
cites explicitly:
Lemma 754 (*$\mathcal{SF}$ disjoint representation*),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*).

The proof of Lemma 765 (*$\mathcal{SF}_+$ canonical representation*)
cites explicitly:
Lemma 752 (*$\mathcal{SF}$ canonical representation*),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*).

The proof of Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*)
cites explicitly:
Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*),
Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*),
Lemma 765 (*$\mathcal{SF}_+$ canonical representation*).

The proof of Lemma 767 ($\mathcal{SF}_+$ *simple representation*)
cites explicitly:
Lemma 749 ($\mathcal{SF}$ *simple representation*),
Definition 763 ($\mathcal{SF}_+$*, subset of nonnegative simple functions*),
Lemma 765 ($\mathcal{SF}_+$ *canonical representation*).

The proof of Lemma 768 ($\mathcal{SF}_+$ *is closed under positive algebra operations*)
cites explicitly:
Definition 226 (*algebra over a field*),
Lemma 757 ($\mathcal{SF}$ *is algebra over* $\mathbb{R}$),
Definition 763 ($\mathcal{SF}_+$*, subset of nonnegative simple functions*),
Lemma 764 ($\mathcal{SF}_+$ *disjoint representation*).

The proof of Lemma 769 ($\mathcal{SF}_+$ *is measurable*)
cites explicitly:
Definition 593 ($\mathcal{M}_+$*, subset of nonnegative measurable numeric functions*),
Definition 732 ($\mathcal{IF}$*, set of measurable indicator functions*),
Lemma 759 ($\mathcal{SF}$ *is measurable*),
Definition 763 ($\mathcal{SF}_+$*, subset of nonnegative simple functions*).

The proof of Lemma 770 (*integral in* $\mathcal{SF}_+$)
cites explicitly:
Lemma 318 (*addition in* $\overline{\mathbb{R}}_+$ *is closed*),
Lemma 338 (*multiplication in* $\overline{\mathbb{R}}_+$ *is closed (measure theory)*),
Definition 611 (*measure*),
Lemma 765 ($\mathcal{SF}_+$ *canonical representation*).

The proof of Lemma 771 (*integral in* $\mathcal{SF}_+$ *generalizes integral in* $\mathcal{IF}$)
cites explicitly:
Definition 732 ($\mathcal{IF}$*, set of measurable indicator functions*),
Lemma 741 (*equivalent definition of integral in* $\mathcal{IF}$),
Lemma 767 ($\mathcal{SF}_+$ *simple representation*),
Lemma 770 (*integral in* $\mathcal{SF}_+$).

The proof of Lemma 772 (*equivalent definition of the integral in* $\mathcal{SF}_+$ *(disjoint)*)
cites explicitly:
Definition 608 (σ*-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 765 ($\mathcal{SF}_+$ *canonical representation*),
Lemma 766 ($\mathcal{SF}_+$ *disjoint representation is subpartition of canonical representation*),
Lemma 770 (*integral in* $\mathcal{SF}_+$).

The proof of Lemma 774 (*integral in* $\mathcal{SF}_+$ *is additive*)
cites explicitly:
Lemma 613 (*measure over countable pseudopartition*),
Lemma 764 ($\mathcal{SF}_+$ *disjoint representation*),
Lemma 768 ($\mathcal{SF}_+$ *is closed under positive algebra operations*),
Lemma 772 (*equivalent definition of the integral in* $\mathcal{SF}_+$ *(disjoint)*).

The proof of Lemma 775 (*decomposition of measure in* $\mathcal{SF}_+$)
cites explicitly:
Lemma 518 (*some Borel subsets*),
Lemma 571 (*inverse image is measurable in* $\mathbb{R}$),
Lemma 613 (*measure over countable pseudopartition*),
Lemma 752 ($\mathcal{SF}$ *canonical representation*),
Lemma 769 ($\mathcal{SF}_+$ *is measurable*).

The proof of Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 571 (*inverse image is measurable in* $\mathbb{R}$),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Definition 611 (*measure*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 769 (*$\mathcal{SF}_+$ is measurable*).

The proof of Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*)
cites explicitly:
Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*),
Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*),
Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 571 (*inverse image is measurable in* $\mathbb{R}$),
Definition 575 (*$\mathcal{M}$, set of measurable numeric functions*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 769 (*$\mathcal{SF}_+$ is measurable*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*).

The proof of Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*)
cites explicitly:
Definition 288 (*multiplication in* $\overline{\mathbb{R}}$),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Definition 611 (*measure*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

The proof of Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*)
cites explicitly:
Lemma 767 (*$\mathcal{SF}_+$ simple representation*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*).

The proof of Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*)
cites explicitly:
Statement(s) from [17],
Definition 226 (*algebra over a field*),
Lemma 757 (*$\mathcal{SF}$ is algebra over* $\mathbb{R}$),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*).

The proof of Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*)
cites explicitly:
Statement(s) from [17],
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*).

The proof of Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*)
cites explicitly:

Lemma 743 (*integral in $\mathcal{IF}$ over subset*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*),
Lemma 761 (*$\mathcal{SF}$ is closed under restriction*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*).

The proof of Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*)
cites explicitly:
Definition 732 (*$\mathcal{IF}$, set of measurable indicator functions*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*).

The proof of Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*)
cites explicitly:
Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*),
Lemma 767 (*$\mathcal{SF}_+$ simple representation*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*).

The proof of Lemma 786 (*integral in $\mathcal{SF}_+$ for counting measure on $\mathbb{N}$*)
cites explicitly:
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*).

The proof of Lemma 787 (*integral in $\mathcal{SF}_+$ for Dirac measure*)
cites explicitly:
Definition 675 (*Dirac measure*),
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*).

The proof of Lemma 789 (*integral in $\mathcal{M}_+$*)
cites explicitly:
Statement(s) from [17],
Lemma 770 (*integral in $\mathcal{SF}_+$*).

The proof of Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*)
cites explicitly:
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*),
Lemma 789 (*integral in $\mathcal{M}_+$*).

The proof of Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*)
cites explicitly:
Lemma 769 (*$\mathcal{SF}_+$ is measurable*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*).

The proof of Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*)
cites explicitly:
Statement(s) from [17],
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*),
Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*),

Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*),
Lemma 789 (*integral in $\mathcal{M}_+$*).

The proof of Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*)
cites explicitly:
Definition 288 (*multiplication in $\overline{\mathbb{R}}$*),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*).

The proof of Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*)
cites explicitly:
Lemma 789 (*integral in $\mathcal{M}_+$*).

The proof of Theorem 796 (*Beppo Levi, monotone convergence*)
cites explicitly:
Statement(s) from [17],
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 578 (*measurability of numeric function*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Definition 616 (*continuity from below*),
Lemma 617 (*measure is continuous from below*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Lemma 759 (*$\mathcal{SF}$ is measurable*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*).

The proof of Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*)
cites explicitly:
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Theorem 796 (*Beppo Levi, monotone convergence*).

The proof of Lemma 799 (*adapted sequence in $\mathcal{M}_+$*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 578 (*measurability of numeric function*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*),
Definition 798 (*adapted sequence*).

The proof of Lemma 800 (*usage of adapted sequences*)
cites explicitly:
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Definition 798 (*adapted sequence*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*).

The proof of Lemma 801 (*integral in $\mathcal{M}_+$ is additive*)
cites explicitly:

Lemma 597 ($\mathcal{M}_+$ *is closed under addition*),
Lemma 779 (*integral in* $\mathcal{SF}_+$ *is positive linear*),
Lemma 799 (*adapted sequence in* $\mathcal{M}_+$),
Lemma 800 (*usage of adapted sequences*).

The proof of Lemma 802 (*integral in* $\mathcal{M}_+$ *is positive linear*)
cites explicitly:
Lemma 792 (*integral in* $\mathcal{M}_+$ *is positive homogeneous*),
Lemma 797 (*integral in* $\mathcal{M}_+$ *is homogeneous at* ∞),
Lemma 801 (*integral in* $\mathcal{M}_+$ *is additive*).

The proof of Lemma 803 (*integral in* $\mathcal{M}_+$ *is* σ*-additive*)
cites explicitly:
Lemma 603 ($\mathcal{M}_+$ *is closed under countable sum*),
Theorem 796 (*Beppo Levi, monotone convergence*).

The proof of Lemma 804 (*integral in* $\mathcal{M}_+$ *of decomposition into nonpositive and nonnegative parts*)
cites explicitly:
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 596 ($\mathcal{M}$ *is closed under absolute value*),
Lemma 801 (*integral in* $\mathcal{M}_+$ *is additive*).

The proof of Lemma 805 (*compatibility of integral in* $\mathcal{M}_+$ *with nonpositive and nonnegative parts*)
cites explicitly:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 801 (*integral in* $\mathcal{M}_+$ *is additive*).

The proof of Lemma 806 (*integral in* $\mathcal{M}_+$ *is almost definite*)
cites explicitly:
Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*),
Lemma 344 (*infinity-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*),
Lemma 578 (*measurability of numeric function*),
Definition 593 ($\mathcal{M}_+$*, subset of nonnegative measurable numeric functions*),
Lemma 599 ($\mathcal{M}_+$ *is closed under nonnegative scalar multiplication*),
Lemma 636 (*negligibility of measurable subset*),
Definition 641 (*property almost satisfied*),
Lemma 791 (*integral in* $\mathcal{M}_+$ *of indicator function*),
Lemma 797 (*integral in* $\mathcal{M}_+$ *is homogeneous at* ∞).

The proof of Lemma 807 (*compatibility of integral in* $\mathcal{M}_+$ *with almost binary relation*)

cites explicitly:
Lemma 283 (*zero is identity element for addition in* $\overline{\mathbb{R}}$),
Lemma 342 (*multiplication in* $\overline{\mathbb{R}}_+$ *is distributive over addition (measure theory)*),
Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*),
Definition 474 (σ*-algebra*),
Definition 516 (*measurable space*),
Lemma 598 ($\mathcal{M}_+$ *is closed under multiplication*),
Definition 611 (*measure*),
Definition 631 (*negligible subset*),
Lemma 636 (*negligibility of measurable subset*),
Definition 641 (*property almost satisfied*),
Lemma 643 (*everywhere implies almost everywhere*),

Definition 650 (*almost binary relation*),
Lemma 657 (*almost equality is equivalence relation*),
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*).

The proof of Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*)
cites explicitly:
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

The proof of Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*)
cites explicitly:
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

The proof of Lemma 810 (*Bienaymé–Chebyshev inequality*)
cites explicitly:
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Lemma 569 (*measurability of indicator function*),
Lemma 578 (*measurability of numeric function*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*).

The proof of Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*)
cites explicitly:
Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*),
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Definition 611 (*measure*),
Lemma 636 (*negligibility of measurable subset*),
Definition 641 (*property almost satisfied*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 810 (*Bienaymé–Chebyshev inequality*).

The proof of Lemma 812 (*bounded by integrable in $\mathcal{M}_+$ is integrable*)
cites explicitly:
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*).

The proof of Lemma 813 (*integral in $\mathcal{M}_+$ over subset*)
cites explicitly:
Lemma 592 (*measurability of restriction*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Lemma 761 (*$\mathcal{SF}$ is closed under restriction*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*),
Definition 798 (*adapted sequence*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 800 (*usage of adapted sequences*).

The proof of Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*)
cites explicitly:
Lemma 569 (*measurability of indicator function*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*),
Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*),
Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

The proof of Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*)
cites explicitly:
Lemma 569 (*measurability of indicator function*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

The proof of Theorem 817 (*Fatou's lemma*)
cites explicitly:
Statement(s) from [17],
Lemma 376 (*infimum of bounded sequence is bounded*),
Lemma 378 (*limit inferior*),
Definition 390 (*pointwise convergence*),
Lemma 392 (*limit inferior bounded from below*),
Lemma 586 (*$\mathcal{M}$ is closed under infimum*),
Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Theorem 796 (*Beppo Levi, monotone convergence*).

The proof of Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*)
cites explicitly:
Statement(s) from [17],
Lemma 391 (*limit inferior and limit superior of pointwise convergent*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Theorem 817 (*Fatou's lemma*).

The proof of Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*)
cites explicitly:
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*),
Lemma 800 (*usage of adapted sequences*).

The proof of Lemma 820 (*integral in $\mathcal{M}_+$ for counting measure on $\mathbb{N}$*)
cites explicitly:
Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*).

The proof of Lemma 822 (*integral in $\mathcal{M}_+$ for Dirac measure*)
cites explicitly:
Definition 675 (*Dirac measure*),
Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*).

The proof of Lemma 824 (*measure of section*)
cites explicitly:
Lemma 551 (*measurability of section*),
Definition 611 (*measure*).

The proof of Lemma 825 (*measure of section of product*)
cites explicitly:
Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*),
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 549 (*section of product*),
Definition 611 (*measure*),
Lemma 824 (*measure of section*).

The proof of Lemma 827 (*measurability of measure of section (finite)*)
cites explicitly:
Definition 217 (*product of subsets of parties*),
Definition 442 (*generated set algebra*),
Lemma 443 (*generated set algebra is minimum*),
Definition 448 (*monotone class*),
Lemma 490 (σ*-algebra contains set algebra*),
Lemma 505 (*set algebra generated by product of* σ*-algebras*),
Lemma 515 (*usage of monotone class theorem*),
Definition 541 (*tensor product of* σ*-algebras*),
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 550 (*compatibility of section with set operations*),
Lemma 551 (*measurability of section*),
Lemma 552 (*countable union of sections is measurable*),
Lemma 553 (*countable intersection of sections is measurable*),
Lemma 599 ($\mathcal{M}_+$ *is closed under nonnegative scalar multiplication*),
Lemma 600 ($\mathcal{M}_+$ *is closed under infimum*),
Lemma 601 ($\mathcal{M}_+$ *is closed under supremum*),
Lemma 603 ($\mathcal{M}_+$ *is closed under countable sum*),
Definition 608 (σ*-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 617 (*measure is continuous from below*),
Lemma 619 (*measure is continuous from above*),
Definition 622 (*finite measure*),
Lemma 623 (*finite measure is bounded*),
Lemma 791 (*integral in* $\mathcal{M}_+$ *of indicator function*),
Lemma 824 (*measure of section*),
Lemma 825 (*measure of section of product*).

The proof of Lemma 828 (*measurability of measure of section*)
cites explicitly:
Lemma 475 (*equivalent definition of* σ*-algebra*),
Lemma 551 (*measurability of section*),
Lemma 601 ($\mathcal{M}_+$ *is closed under supremum*),
Lemma 617 (*measure is continuous from below*),
Definition 622 (*finite measure*),
Lemma 625 (*equivalent definition of* σ*-finite measure*),
Lemma 629 (*restricted measure*),
Lemma 824 (*measure of section*),
Lemma 827 (*measurability of measure of section (finite)*).

The proof of Lemma 831 (*candidate tensor product measure is tensor product measure*)

cites explicitly:
Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*),
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 550 (*compatibility of section with set operations*),
Lemma 551 (*measurability of section*),
Lemma 552 (*countable union of sections is measurable*),
Definition 608 (*σ-additivity over measurable space*),
Definition 611 (*measure*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*),
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*),
Lemma 824 (*measure of section*),
Lemma 825 (*measure of section of product*),
Lemma 828 (*measurability of measure of section*),
Definition 829 (*tensor product measure*),
Definition 830 (*candidate tensor product measure*).

The proof of Lemma 832 (*tensor product of finite measures*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 611 (*measure*),
Definition 622 (*finite measure*),
Definition 829 (*tensor product measure*).

The proof of Lemma 833 (*tensor product of σ-finite measures*)
cites explicitly:
Lemma 542 (*product of measurable subsets is measurable*),
Definition 624 (*σ-finite measure*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Definition 829 (*tensor product measure*).

The proof of Lemma 835 (*uniqueness of tensor product measure (finite)*)
cites explicitly:
Definition 448 (*monotone class*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 490 (*σ-algebra contains set algebra*),
Lemma 505 (*set algebra generated by product of σ-algebras*),
Lemma 515 (*usage of monotone class theorem*),
Definition 541 (*tensor product of σ-algebras*),
Lemma 542 (*product of measurable subsets is measurable*),
Definition 607 (*additivity over measurable space*),
Lemma 617 (*measure is continuous from below*),
Lemma 619 (*measure is continuous from above*),
Lemma 621 (*equivalent definition of measure*),
Lemma 623 (*finite measure is bounded*),
Lemma 627 (*finite measure is σ-finite*),
Definition 829 (*tensor product measure*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 832 (*tensor product of finite measures*).

The proof of Lemma 837 (*uniqueness of tensor product measure*)
cites explicitly:
Lemma 475 (*equivalent definition of σ-algebra*),

Lemma 542 (*product of measurable subsets is measurable*),
Lemma 617 (*measure is continuous from below*),
Definition 622 (*finite measure*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 629 (*restricted measure*),
Lemma 824 (*measure of section*),
Definition 829 (*tensor product measure*),
Definition 830 (*candidate tensor product measure*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 833 (*tensor product of σ-finite measures*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

The proof of Lemma 838 (*negligibility of measurable section*)
cites explicitly:
Lemma 551 (*measurability of section*),
Lemma 636 (*negligibility of measurable subset*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 837 (*uniqueness of tensor product measure*).

The proof of Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$)
cites explicitly:
Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$),
Lemma 728 (*Lebesgue measure is σ-finite*),
Lemma 837 (*uniqueness of tensor product measure*).

The proof of Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*)
cites explicitly:
Lemma 726 (*Lebesgue measure generalizes length of interval*),
Definition 829 (*tensor product measure*),
Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$).

The proof of Lemma 841 (*Lebesgue measure on* $\mathbb{R}^2$ *is zero on lines*)
cites explicitly:
Lemma 343 (*zero-product property in* $\overline{\mathbb{R}}_+$ *(measure theory)*),
Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*).

The proof of Lemma 842 (*Lebesgue measure on* $\mathbb{R}^2$ *is σ-finite*)
cites explicitly:
Lemma 728 (*Lebesgue measure is σ-finite*),
Lemma 833 (*tensor product of σ-finite measures*),
Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$).

The proof of Lemma 843 (*Lebesgue measure on* $\mathbb{R}^2$ *is diffuse*)
cites explicitly:
Definition 626 (*diffuse measure*),
Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*).

The proof of Theorem 846 (*Tonelli*)
cites explicitly:
Lemma 551 (*measurability of section*),
Lemma 554 (*indicator of section*),
Lemma 597 (*$\mathcal{M}_+$ is closed under addition*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Lemma 733 (*indicator and support are each other inverse*),
Lemma 767 (*$\mathcal{SF}_+$ simple representation*),

Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 800 (*usage of adapted sequences*),
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 824 (*measure of section*),
Lemma 828 (*measurability of measure of section*),
Lemma 837 (*uniqueness of tensor product measure*),
Definition 844 (*partial function of function from product space*).

The proof of Lemma 847 (*Tonelli over subset*)
cites explicitly:
Lemma 554 (*indicator of section*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*),
Definition 844 (*partial function of function from product space*),
Theorem 846 (*Tonelli*).

The proof of Lemma 848 (*Tonelli for tensor product*)
cites explicitly:
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Definition 604 (*tensor product of numeric functions*),
Lemma 605 (*measurability of tensor product of numeric functions*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Definition 844 (*partial function of function from product space*),
Theorem 846 (*Tonelli*).

The proof of Lemma 852 (*integrable is measurable*)
cites explicitly:
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Definition 851 (*integrability*).

The proof of Lemma 853 (*equivalent definition of integrability*)
cites explicitly:
Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*),
Lemma 321 (*infinity-sum property in $\overline{\mathbb{R}}_+$*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Definition 851 (*integrability*),
Lemma 852 (*integrable is measurable*).

The proof of Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*)
cites explicitly:
Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*),
Lemma 853 (*equivalent definition of integrability*).

The proof of Lemma 855 (*integrable is almost finite*)
cites explicitly:
Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Lemma 853 (*equivalent definition of integrability*).

The proof of Lemma 856 (*almost bounded by integrable is integrable*)
cites explicitly:

Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 658 (*almost order is order relation*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*),
Lemma 853 (*equivalent definition of integrability*).

The proof of Lemma 857 (*bounded by integrable is integrable*)
cites explicitly:
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*),
Lemma 643 (*everywhere implies almost everywhere*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 856 (*almost bounded by integrable is integrable*).

The proof of Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*),
Definition 851 (*integrability*),
Definition 858 (*integral*).

The proof of Lemma 860 (*integral of zero is zero*)
cites explicitly:
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*).

The proof of Lemma 862 (*compatibility of integral with almost equality*)
cites explicitly:
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*),
Definition 851 (*integrability*),
Definition 858 (*integral*).

The proof of Lemma 864 (*integral over subset*)
cites explicitly:
Lemma 405 (*compatibility of nonpositive and nonnegative parts with mask*),
Lemma 406 (*compatibility of nonpositive and nonnegative parts with restriction*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*),
Definition 851 (*integrability*),
Definition 858 (*integral*).

The proof of Lemma 865 (*integral over subset is σ-additive*)
cites explicitly:
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 853 (*equivalent definition of integrability*),
Definition 858 (*integral*).

The proof of Lemma 866 (*integral over singleton*)
cites explicitly:
Definition 297 (*absolute value in $\overline{\mathbb{R}}$*),
Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*),
Lemma 345 (*finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),

Definition 399 (*nonnegative and nonpositive parts*),
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*),
Definition 851 (*integrability*),
Lemma 853 (*equivalent definition of integrability*),
Definition 858 (*integral*),
Lemma 864 (*integral over subset*).

The proof of Lemma 867 (*integral over interval*)
cites explicitly:
Lemma 518 (*some Borel subsets*),
Definition 626 (*diffuse measure*),
Lemma 865 (*integral over subset is σ-additive*),
Lemma 866 (*integral over singleton*).

The proof of Lemma 868 (*Chasles relation, integral over split intervals*)
cites explicitly:
Lemma 865 (*integral over subset is σ-additive*),
Lemma 867 (*integral over interval*).

The proof of Lemma 869 (*integral for counting measure*)
cites explicitly:
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*),
Lemma 853 (*equivalent definition of integrability*),
Definition 858 (*integral*).

The proof of Lemma 870 (*integral for counting measure on $\mathbb{N}$*)
cites explicitly:
Lemma 869 (*integral for counting measure*).

The proof of Lemma 872 (*integral for Dirac measure*)
cites explicitly:
Definition 675 (*Dirac measure*),
Lemma 869 (*integral for counting measure*).

The proof of Lemma 874 (*seminorm $\mathcal{L}^1$*)
cites explicitly:
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Definition 861 (*merge integral in $\mathcal{M}$ and $\mathcal{M}_+$*).

The proof of Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*)
cites explicitly:
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 874 (*seminorm $\mathcal{L}^1$*).

The proof of Lemma 877 (*compatibility of N_1 with almost equality*)
cites explicitly:
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 874 (*seminorm $\mathcal{L}^1$*).

The proof of Lemma 878 (N_1 *is almost definite*)
cites explicitly:
Lemma 686 (*absolute value is almost definite*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 874 (*seminorm $\mathcal{L}^1$*).

The proof of Lemma 879 (N_1 *is absolutely homogeneous*)
cites explicitly:
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*),
Lemma 874 (*seminorm $\mathcal{L}^1$*).

The proof of Lemma 880 (*integral is homogeneous*)
cites explicitly:
Definition 399 (*nonnegative and nonpositive parts*),
Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 852 (*integrable is measurable*),
Lemma 853 (*equivalent definition of integrability*),
Definition 858 (*integral*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Lemma 879 (N_1 *is absolutely homogeneous*).

The proof of Lemma 882 (*Minkowski inequality in $\mathcal{M}$*)
cites explicitly:
Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 682 (*almost sum*),
Lemma 683 (*compatibility of almost sum with almost equality*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Lemma 877 (*compatibility of N_1 with almost equality*).

The proof of Lemma 883 (*integral is additive*)
cites explicitly:
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 805 (*compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts*),
Lemma 855 (*integrable is almost finite*),
Definition 858 (*integral*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

The proof of Lemma 886 (*equivalent definition of $\mathcal{L}^1$*)
cites explicitly:
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*).

The proof of Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*)
cites explicitly:
Lemma 643 (*everywhere implies almost everywhere*),
Definition 678 (*summability domain*),
Lemma 685 (*almost sum is sum*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*).

The proof of Lemma 888 ($\mathcal{L}^1$ *is seminormed vector space*)
cites explicitly:
Statement(s) from [17],
Definition 237 (*seminorm*),
Lemma 574 (*$\mathcal{M}_{\mathbb{R}}$ is vector space*),
Lemma 643 (*everywhere implies almost everywhere*),
Lemma 878 (*N_1 is almost definite*),
Lemma 879 (*N_1 is absolutely homogeneous*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*),
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*).

The proof of Lemma 890 ($\mathcal{L}^1$ *is closed under absolute value*)
cites explicitly:
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*).

The proof of Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*)
cites explicitly:
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*),
Lemma 301 (*finite absolute value in $\overline{\mathbb{R}}$*),
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 857 (*bounded by integrable is integrable*),
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*).

The proof of Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*)
cites explicitly:
Statement(s) from [17],
Definition 282 (*addition in $\overline{\mathbb{R}}$*),
Lemma 401 (*nonnegative and nonpositive parts are nonnegative*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Definition 851 (*integrability*),
Lemma 853 (*equivalent definition of integrability*),
Definition 858 (*integral*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Lemma 880 (*integral is homogeneous*),
Lemma 883 (*integral is additive*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

The proof of Lemma 893 (*constant function is $\mathcal{L}^1$*)
cites explicitly:
Definition 474 (*σ-algebra*),

Definition 516 (*measurable space*),
Definition 611 (*measure*),
Definition 748 (*$\mathcal{SF}$, vector space of simple functions*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*),
Definition 858 (*integral*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*).

The proof of Lemma 894 (*first mean value theorem*)
cites explicitly:
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*),
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Lemma 893 (*constant function is $\mathcal{L}^1$*).

The proof of Lemma 895 (*variant of first mean value theorem*)
cites explicitly:
Definition 241 (*interval*),
Lemma 894 (*first mean value theorem*).

The proof of Theorem 897 (*Lebesgue, dominated convergence*)
cites explicitly:
Statement(s) from [17],
Definition 237 (*seminorm*),
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*),
Lemma 317 (*absolute value in $\overline{\mathbb{R}}$ is continuous*),
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 391 (*limit inferior and limit superior of pointwise convergent*),
Lemma 392 (*limit inferior bounded from below*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*),
Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*),
Definition 593 (*$\mathcal{M}_+$, subset of nonnegative measurable numeric functions*),
Theorem 817 (*Fatou's lemma*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*),
Definition 889 (*convergence in $\mathcal{L}^1$*),
Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*),
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

The proof of Theorem 899 (*Lebesgue, extended dominated convergence*)
cites explicitly:
Definition 474 (*σ-algebra*),
Lemma 475 (*equivalent definition of σ-algebra*),
Definition 516 (*measurable space*),
Lemma 591 (*measurability and masking*),
Definition 611 (*measure*),
Definition 631 (*negligible subset*),
Lemma 638 (*compatibility of null measure with countable union*),
Definition 641 (*property almost satisfied*),

Lemma 660 (*compatibility of almost equality with operator*),
Lemma 687 (*masking almost nowhere*),
Lemma 688 (*finite nonnegative part*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*),
Theorem 897 (*Lebesgue, dominated convergence*).

Appendix C

Is explicitly cited in the proof of. . .

This appendix gathers the explicit citations that appear in the proof of results (lemmas and theorems) for each statement listed in Appendix A. Statements from [17] are anonymized.

The corresponding dependency graph is represented in Figure A.1 (bottom). The dual graph is described in Appendix B.

Printing is not advised!

Statement(s) from [17]

are explicitly cited in the proof of:
Lemma 221 (*quotient vector space, equivalence relation*),
Lemma 222 (*quotient vector space*),
Lemma 223 (*linear map on quotient vector space*),
Lemma 231 (*algebra of functions to algebra*),
Lemma 235 (*vector subspace and closed under multiplication is subalgebra*),
Lemma 236 (*closed under algebra operations is subalgebra*),
Lemma 239 (*definite seminorm is norm*),
Lemma 269 (*equivalent definition of convergent sequence*),
Lemma 270 (*convergent subsequence of Cauchy sequence*),
Lemma 352 (*connected component of open subset of* $\mathbb{R}$ *is open interval*),
Theorem 355 (*countable connected components of open subsets of* $\mathbb{R}$),
Lemma 358 (*open intervals with rational bounds cover open interval*),
Lemma 363 (*extrema of constant function*),
Lemma 364 (*equivalent definition of finite infimum*),
Lemma 365 (*equivalent definition of finite infimum in* $\overline{\mathbb{R}}$),
Lemma 366 (*equivalent definition of infimum*),
Lemma 367 (*infimum is smaller than supremum*),
Lemma 368 (*infimum is monotone*),
Lemma 369 (*supremum is monotone*),
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 371 (*compatibility of supremum with absolute value*),
Lemma 374 (*infimum of sequence is monotone*),
Lemma 375 (*supremum of sequence is monotone*),
Lemma 379 (*limit inferior is* ∞),
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 392 (*limit inferior bounded from below*),
Lemma 393 (*limit inferior bounded from above*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*),
Lemma 586 ($\mathcal{M}$ *is closed under infimum*),

Lemma 587 (*$\mathcal{M}$ is closed under supremum*),
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 638 (*compatibility of null measure with countable union*),
Lemma 700 (*$\lambda^\star$ is homogeneous*),
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 717 (*rays are Lebesgue-measurable*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*),
Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Theorem 817 (*Fatou's lemma*),
Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*).

Definition 207 (*pseudopartition*)
is explicitly cited in the proof of:
Lemma 209 (*compatibility of pseudopartition with intersection*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 668 (*uniqueness of measures extended from a π-system*).

Lemma 209 (*compatibility of pseudopartition with intersection*)
is explicitly cited in the proof of:
Lemma 536 (*characterization of Borel subsets*),
Lemma 613 (*measure over countable pseudopartition*),
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*).

Lemma 210 (*technical inclusion for countable union*)
is explicitly cited in the proof of:
Lemma 211 (*order is meaningless in countable union*).

Lemma 211 (*order is meaningless in countable union*)
is explicitly cited in the proof of:
Lemma 212 (*definition of double countable union*).

Lemma 212 (*definition of double countable union*)
is explicitly cited in the proof of:
Lemma 213 (*double countable union*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*).

Lemma 213 (*double countable union*)
is explicitly cited in the proof of:
Lemma 702 (*$\lambda^\star$ is σ-subadditive*).

Lemma 215 (*partition of countable union*)
is explicitly cited in the proof of:
Lemma 424 (*closedness under countable disjoint and monotone union*),
Lemma 426 (*closedness under countable disjoint union and countable union*),
Lemma 446 (*partition of countable union in set algebra*),
Lemma 480 (*partition of countable union in σ-algebra*).

Definition 216 (*trace of subsets of parties*)
is explicitly cited in the proof of:
Lemma 260 (*trace topology on subset*),
Lemma 532 (*trace σ-algebra*),
Lemma 533 (*measurability of measurable subspace*),
Lemma 534 (*generating measurable subspace*),
Lemma 537 (*source restriction of measurable function*).

Definition 217 (*product of subsets of parties*)
is explicitly cited in the proof of:
Lemma 261 (*box topology on Cartesian product*),
Lemma 265 (*compatibility of second-countability with Cartesian product*),
Lemma 266 (*complete countable topological basis of product space*),
Lemma 505 (*set algebra generated by product of σ-algebras*),
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 543 (*measurability of function to product space*),
Lemma 546 (*generating product measurable space*),
Lemma 551 (*measurability of section*),
Lemma 827 (*measurability of measure of section (finite)*).

Lemma 218 (*restriction is masking*)
is explicitly cited in the proof of:
Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*),
Lemma 743 (*integral in $\mathcal{IF}$ over subset*).

Definition 219 (*relation compatible with vector operations*)
is explicitly cited in the proof of:
Lemma 220 (*quotient vector operations*),
Lemma 222 (*quotient vector space*).

Lemma 220 (*quotient vector operations*)
is explicitly cited in the proof of:
Lemma 221 (*quotient vector space, equivalence relation*),
Lemma 223 (*linear map on quotient vector space*).

Lemma 221 (*quotient vector space, equivalence relation*)
is explicitly cited in the proof of:
Lemma 222 (*quotient vector space*).

Lemma 222 (*quotient vector space*)
is explicitly cited in the proof of:
Lemma 223 (*linear map on quotient vector space*).

Lemma 223 (*linear map on quotient vector space*)
is not yet used.

Definition 226 (*algebra over a field*)
is explicitly cited in the proof of:
Lemma 228 (*$\mathbb{K}$ is $\mathbb{K}$-algebra*),
Lemma 231 (*algebra of functions to algebra*),
Lemma 235 (*vector subspace and closed under multiplication is subalgebra*),
Lemma 574 (*$\mathcal{M}_{\mathbb{R}}$ is vector space*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*).

Lemma 228 (*$\mathbb{K}$ is $\mathbb{K}$-algebra*)
is explicitly cited in the proof of:

Lemma 232 ($\mathbb{K}^X$ *is algebra*),
Lemma 572 ($\mathcal{M}_\mathbb{R}$ *is algebra*).

Definition 229 (*inherited algebra operations*)
is explicitly cited in the proof of:
Lemma 231 (*algebra of functions to algebra*).

Lemma 231 (*algebra of functions to algebra*)
is explicitly cited in the proof of:
Lemma 232 ($\mathbb{K}^X$ *is algebra*),
Lemma 572 ($\mathcal{M}_\mathbb{R}$ *is algebra*).

Lemma 232 ($\mathbb{K}^X$ *is algebra*)
is not yet used.

Definition 233 (*subalgebra*)
is explicitly cited in the proof of:
Lemma 235 (*vector subspace and closed under multiplication is subalgebra*),
Lemma 574 ($\mathcal{M}_\mathbb{R}$ *is vector space*).

Lemma 235 (*vector subspace and closed under multiplication is subalgebra*)
is explicitly cited in the proof of:
Lemma 236 (*closed under algebra operations is subalgebra*),
Lemma 757 ($\mathcal{SF}$ *is algebra over* $\mathbb{R}$).

Lemma 236 (*closed under algebra operations is subalgebra*)
is explicitly cited in the proof of:
Lemma 572 ($\mathcal{M}_\mathbb{R}$ *is algebra*),
Lemma 574 ($\mathcal{M}_\mathbb{R}$ *is vector space*).

Definition 237 (*seminorm*)
is explicitly cited in the proof of:
Lemma 239 (*definite seminorm is norm*),
Lemma 888 ($\mathcal{L}^1$ *is seminormed vector space*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 239 (*definite seminorm is norm*)
is not yet used.

Definition 241 (*interval*)
is explicitly cited in the proof of:
Lemma 243 (*empty open interval*),
Lemma 246 (*intervals are closed under finite intersection*),
Lemma 352 (*connected component of open subset of* $\mathbb{R}$ *is open interval*),
Lemma 895 (*variant of first mean value theorem*).

Lemma 243 (*empty open interval*)
is explicitly cited in the proof of:
Lemma 247 (*empty intersection of open intervals*).

Lemma 246 (*intervals are closed under finite intersection*)
is explicitly cited in the proof of:
Lemma 247 (*empty intersection of open intervals*),
Lemma 258 (*topological basis of order topology*),
Lemma 717 (*rays are Lebesgue-measurable*),
Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$).

Lemma 247 (*empty intersection of open intervals*)
is not yet used.

Definition 249 (*topological space*)
is explicitly cited in the proof of:
Lemma 250 (*intersection of topologies*),
Lemma 253 (*equivalent definition of generated topology*),
Lemma 260 (*trace topology on subset*),
Lemma 261 (*box topology on Cartesian product*),
Lemma 352 (*connected component of open subset of* $\mathbb{R}$ *is open interval*),
Lemma 358 (*open intervals with rational bounds cover open interval*),
Lemma 518 (*some Borel subsets*).

Lemma 250 (*intersection of topologies*)
is explicitly cited in the proof of:
Lemma 252 (*generated topology is minimum*).

Definition 251 (*generated topology*)
is explicitly cited in the proof of:
Lemma 252 (*generated topology is minimum*).

Lemma 252 (*generated topology is minimum*)
is explicitly cited in the proof of:
Lemma 253 (*equivalent definition of generated topology*).

Lemma 253 (*equivalent definition of generated topology*)
is explicitly cited in the proof of:
Lemma 258 (*topological basis of order topology*).

Definition 254 (*topological basis*)
is explicitly cited in the proof of:
Lemma 255 (*augmented topological basis*),
Lemma 258 (*topological basis of order topology*),
Lemma 260 (*trace topology on subset*),
Lemma 261 (*box topology on Cartesian product*),
Lemma 312 (*trace topology on* $\mathbb{R}$),
Theorem 359 ($\mathbb{R}$ *is second-countable*),
Lemma 362 ($\overline{\mathbb{R}}$ *is second-countable*).

Lemma 255 (*augmented topological basis*)
is explicitly cited in the proof of:
Lemma 264 (*complete countable topological basis*).

Definition 256 (*order topology*)
is explicitly cited in the proof of:
Lemma 258 (*topological basis of order topology*),
Lemma 310 (*topology of* $\overline{\mathbb{R}}$).

Lemma 258 (*topological basis of order topology*)
is explicitly cited in the proof of:
Lemma 310 (*topology of* $\overline{\mathbb{R}}$),
Lemma 312 (*trace topology on* $\mathbb{R}$).

Lemma 260 (*trace topology on subset*)
is explicitly cited in the proof of:
Lemma 312 (*trace topology on* $\mathbb{R}$).

Lemma 261 (*box topology on Cartesian product*)
is explicitly cited in the proof of:
Lemma 265 (*compatibility of second-countability with Cartesian product*),
Lemma 266 (*complete countable topological basis of product space*),
Lemma 360 (*$\mathbb{R}^n$ is second-countable*).

Definition 262 (*second-countability*)
is explicitly cited in the proof of:
Lemma 264 (*complete countable topological basis*),
Theorem 359 (*$\mathbb{R}$ is second-countable*),
Lemma 360 (*$\mathbb{R}^n$ is second-countable*),
Lemma 362 (*$\overline{\mathbb{R}}$ is second-countable*).

Lemma 264 (*complete countable topological basis*)
is explicitly cited in the proof of:
Lemma 266 (*complete countable topological basis of product space*).

Lemma 265 (*compatibility of second-countability with Cartesian product*)
is explicitly cited in the proof of:
Lemma 360 (*$\mathbb{R}^n$ is second-countable*).

Lemma 266 (*complete countable topological basis of product space*)
is explicitly cited in the proof of:
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*).

Definition 267 (*pseudometric*)
is not yet used.

Lemma 269 (*equivalent definition of convergent sequence*)
is explicitly cited in the proof of:
Lemma 364 (*equivalent definition of finite infimum*).

Lemma 270 (*convergent subsequence of Cauchy sequence*)
is not yet used.

Definition 271 (*cluster point*)
is explicitly cited in the proof of:
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 381 (*limit inferior is invariant by translation*).

Lemma 272 (*finite cover of compact interval*)
is explicitly cited in the proof of:
Lemma 703 (*$\lambda^\star$ generalizes length of interval*).

Definition 273 (*Hölder conjugates in $\mathbb{R}$*)
is explicitly cited in the proof of:
Lemma 274 (*2 is self-Hölder conjugate in $\mathbb{R}$*),
Lemma 275 (*Young's inequality for products in $\mathbb{R}$*).

Lemma 274 (*2 is self-Hölder conjugate in $\mathbb{R}$*)
is explicitly cited in the proof of:
Lemma 276 (*Young's inequality for products in $\mathbb{R}$, case $p = 2$*),
Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*).

Lemma 275 (*Young's inequality for products in $\mathbb{R}$*)
is explicitly cited in the proof of:
Lemma 276 (*Young's inequality for products in $\mathbb{R}$, case $p = 2$*),
Lemma 349 (*Young's inequality for products (measure theory)*).

Lemma 276 (*Young's inequality for products in $\mathbb{R}$, case $p = 2$*)
is not yet used.

Definition 278 (*extended real numbers, $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*),
Lemma 287 (*additive inverse in $\overline{\mathbb{R}}$ is monotone*),
Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*),
Lemma 310 (*topology of $\overline{\mathbb{R}}$*),
Lemma 398 (*finite part is finite*),
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*).

Lemma 279 (*order in $\overline{\mathbb{R}}$ is total*)
is explicitly cited in the proof of:
Lemma 367 (*infimum is smaller than supremum*),
Lemma 368 (*infimum is monotone*),
Lemma 374 (*infimum of sequence is monotone*),
Lemma 386 (*limit inferior is smaller than limit superior*),
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 812 (*bounded by integrable in $\mathcal{M}_+$ is integrable*),
Lemma 856 (*almost bounded by integrable is integrable*),
Lemma 857 (*bounded by integrable is integrable*),
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*).

Definition 282 (*addition in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 283 (*zero is identity element for addition in $\overline{\mathbb{R}}$*),
Lemma 284 (*addition in $\overline{\mathbb{R}}$ is associative when defined*),
Lemma 285 (*addition in $\overline{\mathbb{R}}$ is commutative when defined*),
Lemma 286 (*infinity-sum property in $\overline{\mathbb{R}}$*),
Lemma 287 (*additive inverse in $\overline{\mathbb{R}}$ is monotone*),
Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*),
Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*),
Lemma 315 (*continuity of addition in $\overline{\mathbb{R}}$*),
Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*),
Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*),
Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*),
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 614 (*measure is monotone*),
Lemma 679 (*summability on summability domain*),
Lemma 682 (*almost sum*),
Lemma 883 (*integral is additive*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 283 (*zero is identity element for addition in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

Lemma 284 (*addition in $\overline{\mathbb{R}}$ is associative when defined*)
is explicitly cited in the proof of:
Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*).

Lemma 285 (*addition in $\overline{\mathbb{R}}$ is commutative when defined*)
is explicitly cited in the proof of:
Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*).

Lemma 286 (*infinity-sum property in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 321 (*infinity-sum property in $\overline{\mathbb{R}}_+$*).

Lemma 287 (*additive inverse in $\overline{\mathbb{R}}$ is monotone*)
is explicitly cited in the proof of:
Lemma 299 (*bounded absolute value in $\overline{\mathbb{R}}$*),
Lemma 300 (*bounded absolute value in $\overline{\mathbb{R}}$ (strict)*).

Definition 288 (*multiplication in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 290 (*multiplication in $\overline{\mathbb{R}}$ is associative when defined*),
Lemma 291 (*multiplication in $\overline{\mathbb{R}}$ is commutative when defined*),
Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*),
Lemma 294 (*zero-product property in $\overline{\mathbb{R}}$*),
Lemma 295 (*infinity-product property in $\overline{\mathbb{R}}$*),
Lemma 296 (*finite-product property in $\overline{\mathbb{R}}$*),
Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*),
Lemma 316 (*continuity of multiplication in $\overline{\mathbb{R}}$*),
Lemma 329 (*multiplication in $\overline{\mathbb{R}}_+$ is closed when defined*),
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 591 (*measurability and masking*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*).

Lemma 290 (*multiplication in $\overline{\mathbb{R}}$ is associative when defined*)
is explicitly cited in the proof of:
Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*).

Lemma 291 (*multiplication in $\overline{\mathbb{R}}$ is commutative when defined*)
is explicitly cited in the proof of:
Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*),
Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*).

Lemma 292 (*multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined*)
is explicitly cited in the proof of:
Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*),
Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*).

Lemma 293 (*multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined*)
is explicitly cited in the proof of:
Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*).

Lemma 294 (*zero-product property in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 330 (*zero-product property in $\overline{\mathbb{R}}_+$*),
Lemma 335 (*zero-product property in $\overline{\mathbb{R}}$ (measure theory)*).

Lemma 295 (*infinity-product property in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:

Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*),
Lemma 336 (*infinity-product property in $\overline{\mathbb{R}}$ (measure theory)*).

Lemma 296 (*finite-product property in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 332 (*finite-product property in $\overline{\mathbb{R}}_+$*).

Definition 297 (*absolute value in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*),
Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*),
Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*),
Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*),
Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*),
Lemma 317 (*absolute value in $\overline{\mathbb{R}}$ is continuous*),
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 866 (*integral over singleton*).

Lemma 298 (*equivalent definition of absolute value in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 299 (*bounded absolute value in $\overline{\mathbb{R}}$*),
Lemma 300 (*bounded absolute value in $\overline{\mathbb{R}}$ (strict)*),
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*).

Lemma 299 (*bounded absolute value in $\overline{\mathbb{R}}$*)
is not yet used.

Lemma 300 (*bounded absolute value in $\overline{\mathbb{R}}$ (strict)*)
is explicitly cited in the proof of:
Lemma 301 (*finite absolute value in $\overline{\mathbb{R}}$*).

Lemma 301 (*finite absolute value in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*).

Lemma 302 (*absolute value in $\overline{\mathbb{R}}$ is nonnegative*)
is explicitly cited in the proof of:
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 303 (*absolute value in $\overline{\mathbb{R}}$ is even*)
is explicitly cited in the proof of:
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 371 (*compatibility of supremum with absolute value*),
Lemma 389 (*compatibility of limit superior with absolute value*),
Lemma 855 (*integrable is almost finite*).

Lemma 304 (*absolute value in $\overline{\mathbb{R}}$ is definite*)
is explicitly cited in the proof of:
Lemma 686 (*absolute value is almost definite*),
Lemma 866 (*integral over singleton*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 305 (*absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality*)
is explicitly cited in the proof of:
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Definition 306 (*exponential and logarithm in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 307 (*exponential and logarithm in $\overline{\mathbb{R}}$ are inverse*),
Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*).

Lemma 307 (*exponential and logarithm in $\overline{\mathbb{R}}$ are inverse*)
is not yet used.

Definition 308 (*exponentiation in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*),
Lemma 346 (*exponentiation in $\overline{\mathbb{R}}$ (measure theory)*).

Lemma 309 (*exponentiation in $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 346 (*exponentiation in $\overline{\mathbb{R}}$ (measure theory)*),
Lemma 349 (*Young's inequality for products (measure theory)*).

Lemma 310 (*topology of $\overline{\mathbb{R}}$*)
is explicitly cited in the proof of:
Lemma 314 (*convergence towards $-\infty$*),
Lemma 362 (*$\overline{\mathbb{R}}$ is second-countable*).

Lemma 312 (*trace topology on $\mathbb{R}$*)
is not yet used.

Lemma 314 (*convergence towards $-\infty$*)
is explicitly cited in the proof of:
Lemma 366 (*equivalent definition of infimum*).

Lemma 315 (*continuity of addition in $\overline{\mathbb{R}}$*)
is not yet used.

Lemma 316 (*continuity of multiplication in $\overline{\mathbb{R}}$*)
is not yet used.

Lemma 317 (*absolute value in $\overline{\mathbb{R}}$ is continuous*)
is explicitly cited in the proof of:
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 318 (*addition in $\overline{\mathbb{R}}_+$ is closed*)
is explicitly cited in the proof of:
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 597 (*$\mathcal{M}_+$ is closed under addition*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 853 (*equivalent definition of integrability*).

Lemma 319 (*addition in $\overline{\mathbb{R}}_+$ is associative*)
is explicitly cited in the proof of:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 320 (*addition in $\overline{\mathbb{R}}_+$ is commutative*)
is explicitly cited in the proof of:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 708 (*$\mathcal{L}$ is closed under complement*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 321 (*infinity-sum property in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 853 (*equivalent definition of integrability*).

Lemma 322 (*series are convergent in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 323 (*technical upper bound in series in $\overline{\mathbb{R}}_+$*),
Lemma 326 (*double series in $\overline{\mathbb{R}}_+$*),
Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*).

Lemma 323 (*technical upper bound in series in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 324 (*order is meaningless in series in $\overline{\mathbb{R}}_+$*).

Lemma 324 (*order is meaningless in series in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 325 (*definition of double series in $\overline{\mathbb{R}}_+$*).

Lemma 325 (*definition of double series in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 326 (*double series in $\overline{\mathbb{R}}_+$*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*).

Lemma 326 (*double series in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 702 (*$\lambda^\star$ is σ-subadditive*).

Definition 327 (*multiplication in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 329 (*multiplication in $\overline{\mathbb{R}}_+$ is closed when defined*),
Lemma 330 (*zero-product property in $\overline{\mathbb{R}}_+$*),
Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*),
Lemma 332 (*finite-product property in $\overline{\mathbb{R}}_+$*),
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*).

Lemma 329 (*multiplication in $\overline{\mathbb{R}}_+$ is closed when defined*)
is explicitly cited in the proof of:
Lemma 349 (*Young's inequality for products (measure theory)*).

Lemma 330 (*zero-product property in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*).

Lemma 331 (*infinity-product property in $\overline{\mathbb{R}}_+$*)
is explicitly cited in the proof of:
Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

Lemma 332 (*finite-product property in $\overline{\mathbb{R}}_+$*)
is not yet used.

Definition 333 (*multiplication in $\overline{\mathbb{R}}$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 335 (*zero-product property in $\overline{\mathbb{R}}$ (measure theory)*),
Lemma 336 (*infinity-product property in $\overline{\mathbb{R}}$ (measure theory)*),
Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*),
Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*),
Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*),
Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*),
Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*),
Lemma 346 (*exponentiation in $\overline{\mathbb{R}}$ (measure theory)*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

Lemma 335 (*zero-product property in $\overline{\mathbb{R}}$ (measure theory)*)
is not yet used.

Lemma 336 (*infinity-product property in $\overline{\mathbb{R}}$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 337 (*finite-product property in $\overline{\mathbb{R}}$ (measure theory)*).

Lemma 337 (*finite-product property in $\overline{\mathbb{R}}$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 345 (*finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)*).

Lemma 338 (*multiplication in $\overline{\mathbb{R}}_+$ is closed (measure theory)*)
is explicitly cited in the proof of:
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*),
Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 848 (*Tonelli for tensor product*).

Lemma 340 (*multiplication in $\overline{\mathbb{R}}_+$ is associative (measure theory)*)
is explicitly cited in the proof of:
Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*).

Lemma 341 (*multiplication in $\overline{\mathbb{R}}_+$ is commutative (measure theory)*)
is explicitly cited in the proof of:
Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 848 (*Tonelli for tensor product*).

Lemma 342 (*multiplication in $\overline{\mathbb{R}}_+$ is distributive over addition (measure theory)*)
is explicitly cited in the proof of:
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

Lemma 343 (*zero-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),

Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 825 (*measure of section of product*),
Lemma 841 (*Lebesgue measure on $\mathbb{R}^2$ is zero on lines*).

Lemma 344 (*infinity-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 349 (*Young's inequality for products (measure theory)*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*).

Lemma 345 (*finite-product property in $\overline{\mathbb{R}}_+$ (measure theory)*)
is explicitly cited in the proof of:
Lemma 866 (*integral over singleton*).

Lemma 346 (*exponentiation in $\overline{\mathbb{R}}$ (measure theory)*)
is not yet used.

Definition 347 (*Hölder conjugates*)
is not yet used.

Lemma 349 (*Young's inequality for products (measure theory)*)
is explicitly cited in the proof of:
Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*).

Lemma 350 (*Young's inequality for products, case $p = 2$ (measure theory)*)
is not yet used.

Definition 351 (*connected component in $\mathbb{R}$*)
is explicitly cited in the proof of:
Lemma 352 (*connected component of open subset of $\mathbb{R}$ is open interval*),
Lemma 353 (*connected component of open subset of $\mathbb{R}$ is maximal*),
Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*).

Lemma 352 (*connected component of open subset of $\mathbb{R}$ is open interval*)
is explicitly cited in the proof of:
Lemma 353 (*connected component of open subset of $\mathbb{R}$ is maximal*).

Lemma 353 (*connected component of open subset of $\mathbb{R}$ is maximal*)
is explicitly cited in the proof of:
Lemma 354 (*connected components of open subset of $\mathbb{R}$ are equal or disjoint*).

Lemma 354 (*connected components of open subset of $\mathbb{R}$ are equal or disjoint*)
is explicitly cited in the proof of:
Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*).

Theorem 355 (*countable connected components of open subsets of $\mathbb{R}$*)
is explicitly cited in the proof of:
Theorem 359 (*$\mathbb{R}$ is second-countable*),
Lemma 362 (*$\overline{\mathbb{R}}$ is second-countable*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 559 (*countable generator of Borel σ-algebra of $\mathbb{R}$*),
Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*).

Lemma 356 (*rational approximation of lower bound of open interval*)
is explicitly cited in the proof of:
Lemma 357 (*rational approximation of upper bound of open interval*),
Lemma 358 (*open intervals with rational bounds cover open interval*),
Lemma 361 (*open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$*).

Lemma 357 (*rational approximation of upper bound of open interval*)
is explicitly cited in the proof of:
Lemma 358 (*open intervals with rational bounds cover open interval*),
Lemma 361 (*open intervals with rational bounds cover open interval of* $\overline{\mathbb{R}}$).

Lemma 358 (*open intervals with rational bounds cover open interval*)
is explicitly cited in the proof of:
Theorem 359 ($\mathbb{R}$ *is second-countable*),
Lemma 361 (*open intervals with rational bounds cover open interval of* $\overline{\mathbb{R}}$).

Theorem 359 ($\mathbb{R}$ *is second-countable*)
is explicitly cited in the proof of:
Lemma 360 ($\mathbb{R}^n$ *is second-countable*),
Lemma 559 (*countable generator of Borel* σ*-algebra of* $\mathbb{R}$).

Lemma 360 ($\mathbb{R}^n$ *is second-countable*)
is explicitly cited in the proof of:
Lemma 565 (*Borel* σ*-algebra of* $\mathbb{R}^n$).

Lemma 361 (*open intervals with rational bounds cover open interval of* $\overline{\mathbb{R}}$)
is explicitly cited in the proof of:
Lemma 362 ($\overline{\mathbb{R}}$ *is second-countable*).

Lemma 362 ($\overline{\mathbb{R}}$ *is second-countable*)
is not yet used.

Lemma 363 (*extrema of constant function*)
is explicitly cited in the proof of:
Lemma 365 (*equivalent definition of finite infimum in* $\overline{\mathbb{R}}$),
Lemma 376 (*infimum of bounded sequence is bounded*),
Lemma 377 (*supremum of bounded sequence is bounded*).

Lemma 364 (*equivalent definition of finite infimum*)
is explicitly cited in the proof of:
Lemma 365 (*equivalent definition of finite infimum in* $\overline{\mathbb{R}}$).

Lemma 365 (*equivalent definition of finite infimum in* $\overline{\mathbb{R}}$)
is explicitly cited in the proof of:
Lemma 366 (*equivalent definition of infimum*).

Lemma 366 (*equivalent definition of infimum*)
is not yet used.

Lemma 367 (*infimum is smaller than supremum*)
is explicitly cited in the proof of:
Lemma 370 (*compatibility of infimum with absolute value*).

Lemma 368 (*infimum is monotone*)
is explicitly cited in the proof of:
Lemma 369 (*supremum is monotone*),
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 372 (*compatibility of translation with infimum*),
Lemma 378 (*limit inferior*).

Lemma 369 (*supremum is monotone*)
is explicitly cited in the proof of:
Lemma 370 (*compatibility of infimum with absolute value*),
Lemma 373 (*compatibility of translation with supremum*).

Lemma 370 (*compatibility of infimum with absolute value*)
is explicitly cited in the proof of:
Lemma 371 (*compatibility of supremum with absolute value*),
Lemma 388 (*compatibility of limit inferior with absolute value*).

Lemma 371 (*compatibility of supremum with absolute value*)
is not yet used.

Lemma 372 (*compatibility of translation with infimum*)
is not yet used.

Lemma 373 (*compatibility of translation with supremum*)
is not yet used.

Lemma 374 (*infimum of sequence is monotone*)
is explicitly cited in the proof of:
Lemma 375 (*supremum of sequence is monotone*),
Lemma 376 (*infimum of bounded sequence is bounded*),
Lemma 382 (*limit inferior is monotone*).

Lemma 375 (*supremum of sequence is monotone*)
is explicitly cited in the proof of:
Lemma 377 (*supremum of bounded sequence is bounded*),
Lemma 382 (*limit inferior is monotone*).

Lemma 376 (*infimum of bounded sequence is bounded*)
is explicitly cited in the proof of:
Lemma 600 ($\mathcal{M}_+$ *is closed under infimum*),
Theorem 817 (*Fatou's lemma*).

Lemma 377 (*supremum of bounded sequence is bounded*)
is explicitly cited in the proof of:
Lemma 601 ($\mathcal{M}_+$ *is closed under supremum*).

Lemma 378 (*limit inferior*)
is explicitly cited in the proof of:
Lemma 379 (*limit inferior is* ∞),
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 382 (*limit inferior is monotone*),
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 388 (*compatibility of limit inferior with absolute value*),
Lemma 588 ($\mathcal{M}$ *is closed under limit inferior*),
Theorem 817 (*Fatou's lemma*).

Lemma 379 (*limit inferior is* ∞)
is explicitly cited in the proof of:
Lemma 380 (*equivalent definition of the limit inferior*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*).

Lemma 380 (*equivalent definition of the limit inferior*)
is explicitly cited in the proof of:
Lemma 381 (*limit inferior is invariant by translation*),
Lemma 385 (*equivalent definition of limit superior*),
Lemma 386 (*limit inferior is smaller than limit superior*),
Lemma 391 (*limit inferior and limit superior of pointwise convergent*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*).

Lemma 381 (***limit inferior is invariant by translation***)
is explicitly cited in the proof of:
Lemma 382 (*limit inferior is monotone*).

Lemma 382 (***limit inferior is monotone***)
is explicitly cited in the proof of:
Lemma 387 (*limit superior is monotone*),
Lemma 392 (*limit inferior bounded from below*),
Lemma 393 (*limit inferior bounded from above*).

Lemma 383 (***limit superior***)
is explicitly cited in the proof of:
Lemma 384 (*duality limit inferior-limit superior*),
Lemma 388 (*compatibility of limit inferior with absolute value*),
Lemma 589 ($\mathcal{M}$ *is closed under limit superior*).

Lemma 384 (***duality limit inferior-limit superior***)
is explicitly cited in the proof of:
Lemma 385 (*equivalent definition of limit superior*),
Lemma 387 (*limit superior is monotone*),
Lemma 389 (*compatibility of limit superior with absolute value*),
Lemma 394 (*limit superior bounded from below*),
Lemma 395 (*limit superior bounded from above*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 385 (***equivalent definition of limit superior***)
is explicitly cited in the proof of:
Lemma 386 (*limit inferior is smaller than limit superior*),
Lemma 391 (*limit inferior and limit superior of pointwise convergent*),
Lemma 396 (*limit inferior, limit superior and pointwise convergence*).

Lemma 386 (***limit inferior is smaller than limit superior***)
is explicitly cited in the proof of:
Lemma 396 (*limit inferior, limit superior and pointwise convergence*).

Lemma 387 (***limit superior is monotone***)
is not yet used.

Lemma 388 (***compatibility of limit inferior with absolute value***)
is explicitly cited in the proof of:
Lemma 389 (*compatibility of limit superior with absolute value*).

Lemma 389 (***compatibility of limit superior with absolute value***)
is not yet used.

Definition 390 (***pointwise convergence***)
is explicitly cited in the proof of:
Lemma 391 (*limit inferior and limit superior of pointwise convergent*),
Theorem 817 (*Fatou's lemma*).

Lemma 391 (***limit inferior and limit superior of pointwise convergent***)
is explicitly cited in the proof of:
Lemma 392 (*limit inferior bounded from below*),
Lemma 393 (*limit inferior bounded from above*),
Lemma 818 (*integral in* $\mathcal{M}_+$ *of pointwise convergent sequence*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 392 (*limit inferior bounded from below*)
is explicitly cited in the proof of:
Lemma 395 (*limit superior bounded from above*),
Theorem 817 (*Fatou's lemma*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 393 (*limit inferior bounded from above*)
is explicitly cited in the proof of:
Lemma 394 (*limit superior bounded from below*).

Lemma 394 (*limit superior bounded from below*)
is not yet used.

Lemma 395 (*limit superior bounded from above*)
is not yet used.

Lemma 396 (*limit inferior, limit superior and pointwise convergence*)
is explicitly cited in the proof of:
Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*),
Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*),
Theorem 897 (*Lebesgue, dominated convergence*).

Definition 397 (*finite part*)
is explicitly cited in the proof of:
Lemma 398 (*finite part is finite*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 688 (*finite nonnegative part*).

Lemma 398 (*finite part is finite*)
is explicitly cited in the proof of:
Lemma 580 (*$\mathcal{M}$ is closed under finite part*).

Definition 399 (*nonnegative and nonpositive parts*)
is explicitly cited in the proof of:
Lemma 400 (*equivalent definition of nonnegative and nonpositive parts*),
Lemma 401 (*nonnegative and nonpositive parts are nonnegative*),
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 405 (*compatibility of nonpositive and nonnegative parts with mask*),
Lemma 406 (*compatibility of nonpositive and nonnegative parts with restriction*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 866 (*integral over singleton*),
Lemma 880 (*integral is homogeneous*).

Lemma 400 (*equivalent definition of nonnegative and nonpositive parts*)
is explicitly cited in the proof of:
Lemma 688 (*finite nonnegative part*).

Lemma 401 (*nonnegative and nonpositive parts are nonnegative*)
is explicitly cited in the proof of:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 402 (*nonnegative and nonpositive parts are orthogonal*)
is explicitly cited in the proof of:
Lemma 403 (*decomposition into nonnegative and nonpositive parts*),
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*).

Lemma 403 (*decomposition into nonnegative and nonpositive parts*)
is explicitly cited in the proof of:
Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Lemma 866 (*integral over singleton*),
Lemma 869 (*integral for counting measure*).

Lemma 404 (*compatibility of nonpositive and nonnegative parts with addition*)
is explicitly cited in the proof of:
Lemma 805 (*compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts*).

Lemma 405 (*compatibility of nonpositive and nonnegative parts with mask*)
is explicitly cited in the proof of:
Lemma 864 (*integral over subset*).

Lemma 406 (*compatibility of nonpositive and nonnegative parts with restriction*)
is explicitly cited in the proof of:
Lemma 864 (*integral over subset*).

Lemma 408 (*nonempty and with empty or full*)
is explicitly cited in the proof of:
Lemma 438 (*equivalent definition of set algebra*),
Lemma 461 (*other properties of λ-system*),
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 488 (*π-system contains σ-algebra*).

Lemma 409 (*with empty and full*)
is explicitly cited in the proof of:
Lemma 422 (*closedness under countable disjoint union and local complement*),
Lemma 438 (*equivalent definition of set algebra*),
Lemma 461 (*other properties of λ-system*),
Lemma 475 (*equivalent definition of σ-algebra*).

Lemma 411 (*closedness under local complement and complement*)
is explicitly cited in the proof of:
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 460 (*equivalent definition of λ-system*).

Lemma 412 (*closedness under disjoint union and local complement*)
is explicitly cited in the proof of:
Lemma 422 (*closedness under countable disjoint union and local complement*).

Lemma 413 (*closedness under set difference and local complement*)
is explicitly cited in the proof of:
Lemma 427 (*closedness under countable monotone union and countable union*),
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 440 (*set algebra is closed under local complement*).

Lemma 414 (*closedness under intersection and set difference*)
is explicitly cited in the proof of:
Lemma 416 (*closedness under union and set difference*),

Lemma 426 (*closedness under countable disjoint union and countable union*),
Lemma 427 (*closedness under countable monotone union and countable union*),
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 711 (*$\mathcal{L}$ is set algebra*).

Lemma 415 (*closedness under union and intersection*)
is explicitly cited in the proof of:
Lemma 416 (*closedness under union and set difference*),
Lemma 418 (*closedness under finite union and intersection*),
Lemma 426 (*closedness under countable disjoint union and countable union*),
Lemma 427 (*closedness under countable monotone union and countable union*).

Lemma 416 (*closedness under union and set difference*)
is not yet used.

Lemma 417 (*closedness under finite operations*)
is explicitly cited in the proof of:
Lemma 418 (*closedness under finite union and intersection*),
Lemma 422 (*closedness under countable disjoint union and local complement*),
Lemma 426 (*closedness under countable disjoint union and countable union*),
Lemma 438 (*equivalent definition of set algebra*),
Lemma 447 (*explicit set algebra*),
Lemma 472 (*λ-system generated by π-system*).

Lemma 418 (*closedness under finite union and intersection*)
is explicitly cited in the proof of:
Lemma 438 (*equivalent definition of set algebra*).

Lemma 421 (*closedness under countable and finite disjoint union*)
is explicitly cited in the proof of:
Lemma 422 (*closedness under countable disjoint union and local complement*).

Lemma 422 (*closedness under countable disjoint union and local complement*)
is explicitly cited in the proof of:
Lemma 460 (*equivalent definition of λ-system*).

Lemma 423 (*closedness under countable union and intersection*)
is explicitly cited in the proof of:
Lemma 461 (*other properties of λ-system*),
Lemma 475 (*equivalent definition of σ-algebra*).

Lemma 424 (*closedness under countable disjoint and monotone union*)
is explicitly cited in the proof of:
Lemma 460 (*equivalent definition of λ-system*).

Lemma 425 (*closedness under countable monotone and disjoint union*)
is explicitly cited in the proof of:
Lemma 427 (*closedness under countable monotone union and countable union*),
Lemma 460 (*equivalent definition of λ-system*).

Lemma 426 (*closedness under countable disjoint union and countable union*)
is explicitly cited in the proof of:
Lemma 427 (*closedness under countable monotone union and countable union*),
Lemma 488 (*π-system contains σ-algebra*),
Lemma 499 (*λ-system contains σ-algebra*).

Lemma 427 (*closedness under countable monotone union and countable union*)
is explicitly cited in the proof of:
Lemma 491 (*set algebra contains σ-algebra*),
Lemma 495 (*monotone class contains σ-algebra*).

Definition 428 (*π-system*)
is explicitly cited in the proof of:
Lemma 431 (*intersection of π-systems*),
Lemma 472 (*λ-system generated by π-system*),
Lemma 486 (*σ-algebra is π-system*),
Lemma 488 (*π-system contains σ-algebra*),
Lemma 506 (*π-system and λ-system is σ-algebra*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

Lemma 431 (*intersection of π-systems*)
is explicitly cited in the proof of:
Lemma 433 (*generated π-system is minimum*).

Definition 432 (*generated π-system*)
is explicitly cited in the proof of:
Lemma 433 (*generated π-system is minimum*).

Lemma 433 (*generated π-system is minimum*)
is explicitly cited in the proof of:
Lemma 434 (*π-system generation is monotone*),
Lemma 435 (*π-system generation is idempotent*),
Lemma 487 (*σ-algebra contains π-system*),
Lemma 488 (*π-system contains σ-algebra*),
Lemma 489 (*σ-algebra generated by π-system*),
Lemma 510 (*usage of Dynkin π–λ theorem*).

Lemma 434 (*π-system generation is monotone*)
is not yet used.

Lemma 435 (*π-system generation is idempotent*)
is explicitly cited in the proof of:
Lemma 668 (*uniqueness of measures extended from a π-system*).

Definition 437 (*set algebra*)
is explicitly cited in the proof of:
Lemma 438 (*equivalent definition of set algebra*),
Lemma 441 (*intersection of set algebras*),
Lemma 447 (*explicit set algebra*),
Lemma 477 (*σ-algebra is set algebra*),
Lemma 491 (*set algebra contains σ-algebra*),
Lemma 511 (*algebra and monotone class is σ-algebra*),
Lemma 719 (*$\mathcal{L}$ is σ-algebra*).

Lemma 438 (*equivalent definition of set algebra*)
is explicitly cited in the proof of:
Lemma 439 (*other equivalent definition of set algebra*),
Lemma 446 (*partition of countable union in set algebra*),
Lemma 447 (*explicit set algebra*).

Lemma 439 (*other equivalent definition of set algebra*)
is explicitly cited in the proof of:

Lemma 440 (*set algebra is closed under local complement*),
Lemma 446 (*partition of countable union in set algebra*),
Lemma 458 (*monotone class generated by set algebra*),
Lemma 478 (*σ-algebra is closed under set difference*),
Lemma 711 (*$\mathcal{L}$ is set algebra*).

Lemma 440 (*set algebra is closed under local complement*)
is explicitly cited in the proof of:
Lemma 478 (*σ-algebra is closed under set difference*).

Lemma 441 (*intersection of set algebras*)
is explicitly cited in the proof of:
Lemma 443 (*generated set algebra is minimum*).

Definition 442 (*generated set algebra*)
is explicitly cited in the proof of:
Lemma 443 (*generated set algebra is minimum*),
Lemma 827 (*measurability of measure of section (finite)*).

Lemma 443 (*generated set algebra is minimum*)
is explicitly cited in the proof of:
Lemma 444 (*set algebra generation is monotone*),
Lemma 445 (*set algebra generation is idempotent*),
Lemma 447 (*explicit set algebra*),
Lemma 490 (*σ-algebra contains set algebra*),
Lemma 491 (*set algebra contains σ-algebra*),
Lemma 492 (*σ-algebra generated by set algebra*),
Lemma 515 (*usage of monotone class theorem*),
Lemma 827 (*measurability of measure of section (finite)*).

Lemma 444 (*set algebra generation is monotone*)
is not yet used.

Lemma 445 (*set algebra generation is idempotent*)
is not yet used.

Lemma 446 (*partition of countable union in set algebra*)
is explicitly cited in the proof of:
Lemma 480 (*partition of countable union in σ-algebra*),
Lemma 715 (*partition of countable union in $\mathcal{L}$*).

Lemma 447 (*explicit set algebra*)
is explicitly cited in the proof of:
Lemma 505 (*set algebra generated by product of σ-algebras*).

Definition 448 (*monotone class*)
is explicitly cited in the proof of:
Lemma 449 (*intersection of monotone classes*),
Lemma 456 (*$\mathcal{C}^{\setminus}$ is monotone class*),
Lemma 493 (*σ-algebra is monotone class*),
Lemma 495 (*monotone class contains σ-algebra*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 449 (*intersection of monotone classes*)
is explicitly cited in the proof of:
Lemma 451 (*generated monotone class is minimum*).

Definition 450 (***generated monotone class***)
is explicitly cited in the proof of:
Lemma 451 (*generated monotone class is minimum*).

Lemma 451 (***generated monotone class is minimum***)
is explicitly cited in the proof of:
Lemma 452 (*monotone class generation is monotone*),
Lemma 453 (*monotone class generation is idempotent*),
Lemma 457 (*monotone class is closed under set difference*),
Lemma 458 (*monotone class generated by set algebra*),
Lemma 494 (*σ-algebra contains monotone class*),
Lemma 495 (*monotone class contains σ-algebra*),
Lemma 496 (*σ-algebra generated by monotone class*),
Theorem 513 (*monotone class*).

Lemma 452 (***monotone class generation is monotone***)
is explicitly cited in the proof of:
Lemma 515 (*usage of monotone class theorem*).

Lemma 453 (***monotone class generation is idempotent***)
is explicitly cited in the proof of:
Lemma 511 (*algebra and monotone class is σ-algebra*),
Lemma 515 (*usage of monotone class theorem*).

Definition 454 (***monotone class and symmetric set difference***)
is explicitly cited in the proof of:
Lemma 455 (*$\mathcal{C}^\setminus$ is symmetric*),
Lemma 456 (*$\mathcal{C}^\setminus$ is monotone class*),
Lemma 457 (*monotone class is closed under set difference*).

Lemma 455 (***$\mathcal{C}^\setminus$ is symmetric***)
is explicitly cited in the proof of:
Lemma 457 (*monotone class is closed under set difference*).

Lemma 456 (***$\mathcal{C}^\setminus$ is monotone class***)
is explicitly cited in the proof of:
Lemma 457 (*monotone class is closed under set difference*).

Lemma 457 (***monotone class is closed under set difference***)
is explicitly cited in the proof of:
Lemma 458 (*monotone class generated by set algebra*).

Lemma 458 (***monotone class generated by set algebra***)
is explicitly cited in the proof of:
Theorem 513 (*monotone class*).

Definition 459 (***λ-system***)
is explicitly cited in the proof of:
Lemma 460 (*equivalent definition of λ-system*),
Lemma 461 (*other properties of λ-system*),
Lemma 462 (*intersection of λ-systems*),
Lemma 469 (*$\Lambda^\cap$ is λ-system*),
Lemma 497 (*σ-algebra is λ-system*),
Lemma 499 (*λ-system contains σ-algebra*).

Lemma 460 (***equivalent definition of λ-system***)
is explicitly cited in the proof of:

Lemma 461 (*other properties of λ-system*),
Lemma 469 (*$\Lambda^\cap$ is λ-system*),
Lemma 668 (*uniqueness of measures extended from a π-system*).

Lemma 461 (*other properties of λ-system*)
is not yet used.

Lemma 462 (*intersection of λ-systems*)
is explicitly cited in the proof of:
Lemma 464 (*generated λ-system is minimum*).

Definition 463 (*generated λ-system*)
is explicitly cited in the proof of:
Lemma 464 (*generated λ-system is minimum*).

Lemma 464 (*generated λ-system is minimum*)
is explicitly cited in the proof of:
Lemma 465 (*λ-system generation is monotone*),
Lemma 466 (*λ-system generation is idempotent*),
Lemma 470 (*λ-system with intersection*),
Lemma 472 (*λ-system generated by π-system*),
Lemma 498 (*σ-algebra contains λ-system*),
Lemma 499 (*λ-system contains σ-algebra*),
Lemma 500 (*σ-algebra generated by λ-system*),
Theorem 508 (*Dynkin π–λ theorem*).

Lemma 465 (*λ-system generation is monotone*)
is explicitly cited in the proof of:
Lemma 510 (*usage of Dynkin π–λ theorem*).

Lemma 466 (*λ-system generation is idempotent*)
is explicitly cited in the proof of:
Lemma 506 (*π-system and λ-system is σ-algebra*),
Lemma 510 (*usage of Dynkin π–λ theorem*).

Definition 467 (*λ-system and intersection*)
is explicitly cited in the proof of:
Lemma 468 (*$\Lambda^\cap$ is symmetric*),
Lemma 470 (*λ-system with intersection*),
Lemma 471 (*λ-system is closed under intersection*).

Lemma 468 (*$\Lambda^\cap$ is symmetric*)
is explicitly cited in the proof of:
Lemma 470 (*λ-system with intersection*).

Lemma 469 (*$\Lambda^\cap$ is λ-system*)
is explicitly cited in the proof of:
Lemma 470 (*λ-system with intersection*).

Lemma 470 (*λ-system with intersection*)
is explicitly cited in the proof of:
Lemma 471 (*λ-system is closed under intersection*).

Lemma 471 (*λ-system is closed under intersection*)
is explicitly cited in the proof of:
Lemma 472 (*λ-system generated by π-system*).

Lemma 472 (*λ-system generated by π-system*)
is explicitly cited in the proof of:
Theorem 508 (*Dynkin π–λ theorem*).

Definition 474 (*σ-algebra*)
is explicitly cited in the proof of:
Lemma 475 (*equivalent definition of σ-algebra*),
Lemma 477 (*σ-algebra is set algebra*),
Lemma 480 (*partition of countable union in σ-algebra*),
Lemma 481 (*intersection of σ-algebras*),
Lemma 488 (*π-system contains σ-algebra*),
Lemma 491 (*set algebra contains σ-algebra*),
Lemma 495 (*monotone class contains σ-algebra*),
Lemma 503 (*countable σ-algebra generator*),
Lemma 518 (*some Borel subsets*),
Lemma 523 (*inverse σ-algebra*),
Lemma 524 (*image σ-algebra*),
Lemma 532 (*trace σ-algebra*),
Lemma 536 (*characterization of Borel subsets*),
Lemma 551 (*measurability of section*),
Lemma 552 (*countable union of sections is measurable*),
Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$),
Lemma 561 (*Borel subsets of* $\overline{\mathbb{R}}$ *and* $\mathbb{R}$),
Lemma 563 (*Borel σ-algebra of* $\mathbb{R}_+$),
Lemma 569 (*measurability of indicator function*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 591 (*measurability and masking*),
Lemma 592 (*measurability of restriction*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 621 (*equivalent definition of measure*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 637 (*empty set is negligible*),
Lemma 638 (*compatibility of null measure with countable union*),
Lemma 671 (*counting measure*),
Lemma 673 (*σ-finite counting measure*),
Lemma 687 (*masking almost nowhere*),
Lemma 688 (*finite nonnegative part*),
Lemma 719 (*$\mathcal{L}$ is σ-algebra*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 893 (*constant function is* $\mathcal{L}^1$),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 475 (*equivalent definition of σ-algebra*)
is explicitly cited in the proof of:
Lemma 479 (*other properties of σ-algebra*),
Lemma 486 (*σ-algebra is π-system*),
Lemma 497 (*σ-algebra is λ-system*),
Lemma 499 (*λ-system contains σ-algebra*),
Lemma 505 (*set algebra generated by product of σ-algebras*),
Lemma 526 (*constant function is measurable*),
Lemma 533 (*measurability of measurable subspace*),
Lemma 536 (*characterization of Borel subsets*),

Lemma 539 (*measurability of function defined on a pseudopartition*),
Lemma 543 (*measurability of function to product space*),
Lemma 553 (*countable intersection of sections is measurable*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*),
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*),
Lemma 569 (*measurability of indicator function*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 586 (*$\mathcal{M}$ is closed under infimum*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*),
Lemma 592 (*measurability of restriction*),
Lemma 613 (*measure over countable pseudopartition*),
Lemma 619 (*measure is continuous from above*),
Lemma 623 (*finite measure is bounded*),
Lemma 629 (*restricted measure*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Lemma 672 (*finiteness of counting measure*),
Lemma 680 (*measurability of summability domain*),
Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*),
Lemma 752 (*$\mathcal{SF}$ canonical representation*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 828 (*measurability of measure of section*),
Lemma 832 (*tensor product of finite measures*),
Lemma 835 (*uniqueness of tensor product measure (finite)*),
Lemma 837 (*uniqueness of tensor product measure*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 477 (*σ-algebra is set algebra*)
is explicitly cited in the proof of:
Lemma 478 (*σ-algebra is closed under set difference*),
Lemma 480 (*partition of countable union in σ-algebra*),
Lemma 490 (*σ-algebra contains set algebra*).

Lemma 478 (*σ-algebra is closed under set difference*)
is explicitly cited in the proof of:
Lemma 614 (*measure is monotone*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 619 (*measure is continuous from above*).

Lemma 479 (*other properties of σ-algebra*)
is explicitly cited in the proof of:
Lemma 493 (*σ-algebra is monotone class*),
Lemma 497 (*σ-algebra is λ-system*).

Lemma 480 (*partition of countable union in σ-algebra*)
is explicitly cited in the proof of:
Lemma 617 (*measure is continuous from below*).

Lemma 481 (*intersection of σ-algebras*)
is explicitly cited in the proof of:
Lemma 483 (*generated σ-algebra is minimum*).

Definition 482 (*generated σ-algebra*)
is explicitly cited in the proof of:
Lemma 483 (*generated σ-algebra is minimum*),
Lemma 510 (*usage of Dynkin π–λ theorem*),
Lemma 515 (*usage of monotone class theorem*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*).

Lemma 483 (*generated σ-algebra is minimum*)
is explicitly cited in the proof of:
Lemma 484 (*σ-algebra generation is monotone*),
Lemma 485 (*σ-algebra generation is idempotent*),
Lemma 487 (*σ-algebra contains π-system*),
Lemma 488 (*π-system contains σ-algebra*),
Lemma 489 (*σ-algebra generated by π-system*),
Lemma 490 (*σ-algebra contains set algebra*),
Lemma 491 (*set algebra contains σ-algebra*),
Lemma 492 (*σ-algebra generated by set algebra*),
Lemma 494 (*σ-algebra contains monotone class*),
Lemma 495 (*monotone class contains σ-algebra*),
Lemma 496 (*σ-algebra generated by monotone class*),
Lemma 498 (*σ-algebra contains λ-system*),
Lemma 499 (*λ-system contains σ-algebra*),
Lemma 500 (*σ-algebra generated by λ-system*),
Lemma 502 (*complete generated σ-algebra*),
Lemma 503 (*countable σ-algebra generator*),
Lemma 518 (*some Borel subsets*),
Lemma 527 (*inverse image of generating family*),
Lemma 529 (*continuous is measurable*),
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 546 (*generating product measurable space*),
Lemma 551 (*measurability of section*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

Lemma 484 (*σ-algebra generation is monotone*)
is explicitly cited in the proof of:
Lemma 489 (*σ-algebra generated by π-system*),
Lemma 492 (*σ-algebra generated by set algebra*),
Lemma 496 (*σ-algebra generated by monotone class*),
Lemma 500 (*σ-algebra generated by λ-system*),
Lemma 501 (*other σ-algebra generator*),
Lemma 502 (*complete generated σ-algebra*),
Lemma 510 (*usage of Dynkin π–λ theorem*),
Lemma 515 (*usage of monotone class theorem*),
Lemma 527 (*inverse image of generating family*),
Lemma 528 (*equivalent definition of measurable function*),
Lemma 546 (*generating product measurable space*),
Lemma 721 (*$\mathcal{B}(\mathbb{R})$ is sub-σ-algebra of $\mathcal{L}$*).

Lemma 485 (*σ-algebra generation is idempotent*)
is explicitly cited in the proof of:
Lemma 501 (*other σ-algebra generator*),
Lemma 506 (*π-system and λ-system is σ-algebra*),
Theorem 508 (*Dynkin π–λ theorem*),

Lemma 511 (*algebra and monotone class is σ-algebra*),
Theorem 513 (*monotone class*),
Lemma 527 (*inverse image of generating family*),
Lemma 528 (*equivalent definition of measurable function*).

Lemma 486 (*σ-algebra is π-system*)
is explicitly cited in the proof of:
Lemma 487 (*σ-algebra contains π-system*).

Lemma 487 (*σ-algebra contains π-system*)
is explicitly cited in the proof of:
Lemma 489 (*σ-algebra generated by π-system*).

Lemma 488 (*π-system contains σ-algebra*)
is not yet used.

Lemma 489 (*σ-algebra generated by π-system*)
is not yet used.

Lemma 490 (*σ-algebra contains set algebra*)
is explicitly cited in the proof of:
Lemma 492 (*σ-algebra generated by set algebra*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 491 (*set algebra contains σ-algebra*)
is not yet used.

Lemma 492 (*σ-algebra generated by set algebra*)
is not yet used.

Lemma 493 (*σ-algebra is monotone class*)
is explicitly cited in the proof of:
Lemma 494 (*σ-algebra contains monotone class*).

Lemma 494 (*σ-algebra contains monotone class*)
is explicitly cited in the proof of:
Lemma 496 (*σ-algebra generated by monotone class*),
Lemma 511 (*algebra and monotone class is σ-algebra*).

Lemma 495 (*monotone class contains σ-algebra*)
is explicitly cited in the proof of:
Lemma 511 (*algebra and monotone class is σ-algebra*).

Lemma 496 (*σ-algebra generated by monotone class*)
is explicitly cited in the proof of:
Theorem 513 (*monotone class*).

Lemma 497 (*σ-algebra is λ-system*)
is explicitly cited in the proof of:
Lemma 498 (*σ-algebra contains λ-system*).

Lemma 498 (*σ-algebra contains λ-system*)
is explicitly cited in the proof of:
Lemma 500 (*σ-algebra generated by λ-system*),
Lemma 506 (*π-system and λ-system is σ-algebra*).

Lemma 499 (*λ-system contains σ-algebra*)
is explicitly cited in the proof of:
Lemma 506 (*π-system and λ-system is σ-algebra*).

Lemma 500 (*σ-algebra generated by λ-system*)
is explicitly cited in the proof of:
Theorem 508 (*Dynkin π–λ theorem*).

Lemma 501 (*other σ-algebra generator*)
is explicitly cited in the proof of:
Lemma 502 (*complete generated σ-algebra*),
Lemma 503 (*countable σ-algebra generator*),
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*).

Lemma 502 (*complete generated σ-algebra*)
is explicitly cited in the proof of:
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*).

Lemma 503 (*countable σ-algebra generator*)
is explicitly cited in the proof of:
Lemma 519 (*countable Borel σ-algebra generator*).

Lemma 505 (*set algebra generated by product of σ-algebras*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 506 (*π-system and λ-system is σ-algebra*)
is explicitly cited in the proof of:
Theorem 508 (*Dynkin π–λ theorem*).

Theorem 508 (*Dynkin π–λ theorem*)
is explicitly cited in the proof of:
Lemma 510 (*usage of Dynkin π–λ theorem*).

Lemma 510 (*usage of Dynkin π–λ theorem*)
is explicitly cited in the proof of:
Lemma 668 (*uniqueness of measures extended from a π-system*).

Lemma 511 (*algebra and monotone class is σ-algebra*)
is explicitly cited in the proof of:
Theorem 513 (*monotone class*).

Theorem 513 (*monotone class*)
is explicitly cited in the proof of:
Lemma 515 (*usage of monotone class theorem*).

Lemma 515 (*usage of monotone class theorem*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Definition 516 (*measurable space*)
is explicitly cited in the proof of:
Lemma 523 (*inverse σ-algebra*),
Lemma 524 (*image σ-algebra*),
Lemma 532 (*trace σ-algebra*),

Lemma 551 (*measurability of section*),
Lemma 552 (*countable union of sections is measurable*),
Lemma 569 (*measurability of indicator function*),
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 591 (*measurability and masking*),
Lemma 592 (*measurability of restriction*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 621 (*equivalent definition of measure*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 637 (*empty set is negligible*),
Lemma 638 (*compatibility of null measure with countable union*),
Lemma 671 (*counting measure*),
Lemma 673 (*σ-finite counting measure*),
Lemma 687 (*masking almost nowhere*),
Lemma 688 (*finite nonnegative part*),
Lemma 735 (*$\mathcal{IF}$ is σ-additive*),
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 893 (*constant function is $\mathcal{L}^1$*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Definition 517 (*Borel σ-algebra*)
is explicitly cited in the proof of:
Lemma 518 (*some Borel subsets*),
Lemma 519 (*countable Borel σ-algebra generator*),
Lemma 529 (*continuous is measurable*),
Lemma 535 (*Borel sub-σ-algebra*),
Lemma 536 (*characterization of Borel subsets*),
Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*),
Lemma 570 (*measurability of numeric function to $\mathbb{R}$*),
Lemma 578 (*measurability of numeric function*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

Lemma 518 (*some Borel subsets*)
is explicitly cited in the proof of:
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 570 (*measurability of numeric function to $\mathbb{R}$*),
Lemma 571 (*inverse image is measurable in $\mathbb{R}$*),
Lemma 578 (*measurability of numeric function*),
Lemma 579 (*inverse image is measurable*),
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*),
Lemma 867 (*integral over interval*).

Lemma 519 (*countable Borel σ-algebra generator*)
is explicitly cited in the proof of:
Lemma 558 (*Borel σ-algebra of $\mathbb{R}$*),
Lemma 560 (*Borel σ-algebra of $\overline{\mathbb{R}}$*),
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*).

Definition 522 (*measurable function*)
is explicitly cited in the proof of:
Lemma 523 (*inverse σ-algebra*),
Lemma 524 (*image σ-algebra*),

Lemma 525 (*identity function is measurable*),
Lemma 528 (*equivalent definition of measurable function*),
Lemma 529 (*continuous is measurable*),
Lemma 530 (*compatibility of measurability with composition*),
Lemma 532 (*trace σ-algebra*),
Lemma 537 (*source restriction of measurable function*),
Lemma 538 (*destination restriction of measurable function*),
Lemma 539 (*measurability of function defined on a pseudopartition*),
Lemma 543 (*measurability of function to product space*),
Lemma 544 (*canonical projection is measurable*),
Lemma 546 (*generating product measurable space*),
Lemma 555 (*measurability of function from product space*),
Lemma 570 (*measurability of numeric function to* $\mathbb{R}$),
Lemma 571 (*inverse image is measurable in* $\mathbb{R}$),
Lemma 577 ($\mathcal{M}$ *and finite is* $\mathcal{M}_\mathbb{R}$),
Lemma 578 (*measurability of numeric function*),
Lemma 579 (*inverse image is measurable*),
Lemma 581 ($\mathcal{M}$ *is closed under addition when defined*),
Lemma 583 ($\mathcal{M}$ *is closed under multiplication*),
Lemma 592 (*measurability of restriction*).

Lemma 523 (*inverse σ-algebra*)
is explicitly cited in the proof of:
Lemma 527 (*inverse image of generating family*).

Lemma 524 (*image σ-algebra*)
is explicitly cited in the proof of:
Lemma 527 (*inverse image of generating family*).

Lemma 525 (*identity function is measurable*)
is not yet used.

Lemma 526 (*constant function is measurable*)
is explicitly cited in the proof of:
Lemma 572 ($\mathcal{M}_\mathbb{R}$ *is algebra*),
Lemma 580 ($\mathcal{M}$ *is closed under finite part*),
Lemma 581 ($\mathcal{M}$ *is closed under addition when defined*),
Lemma 583 ($\mathcal{M}$ *is closed under multiplication*),
Lemma 585 ($\mathcal{M}$ *is closed under scalar multiplication*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*).

Lemma 527 (*inverse image of generating family*)
is explicitly cited in the proof of:
Lemma 528 (*equivalent definition of measurable function*),
Lemma 534 (*generating measurable subspace*),
Lemma 546 (*generating product measurable space*).

Lemma 528 (*equivalent definition of measurable function*)
is explicitly cited in the proof of:
Lemma 529 (*continuous is measurable*),
Lemma 543 (*measurability of function to product space*),
Lemma 570 (*measurability of numeric function to* $\mathbb{R}$),
Lemma 578 (*measurability of numeric function*).

Lemma 529 (*continuous is measurable*)
is explicitly cited in the proof of:

Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*).

Lemma 530 (*compatibility of measurability with composition*)
is explicitly cited in the proof of:
Lemma 537 (*source restriction of measurable function*),
Lemma 572 (*$\mathcal{M}_{\mathbb{R}}$ is algebra*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 605 (*measurability of tensor product of numeric functions*).

Lemma 532 (*trace σ-algebra*)
is explicitly cited in the proof of:
Lemma 533 (*measurability of measurable subspace*),
Lemma 537 (*source restriction of measurable function*),
Lemma 628 (*trace measure*).

Lemma 533 (*measurability of measurable subspace*)
is explicitly cited in the proof of:
Lemma 535 (*Borel sub-σ-algebra*),
Lemma 628 (*trace measure*),
Lemma 739 (*$\mathcal{IF}$ is closed under restriction*).

Lemma 534 (*generating measurable subspace*)
is explicitly cited in the proof of:
Lemma 535 (*Borel sub-σ-algebra*),
Lemma 564 (*Borel σ-algebra of $\overline{\mathbb{R}}_+$*).

Lemma 535 (*Borel sub-σ-algebra*)
is explicitly cited in the proof of:
Lemma 536 (*characterization of Borel subsets*),
Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*),
Lemma 563 (*Borel σ-algebra of $\mathbb{R}_+$*).

Lemma 536 (*characterization of Borel subsets*)
is explicitly cited in the proof of:
Lemma 561 (*Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$*).

Lemma 537 (*source restriction of measurable function*)
is not yet used.

Lemma 538 (*destination restriction of measurable function*)
is not yet used.

Lemma 539 (*measurability of function defined on a pseudopartition*)
is explicitly cited in the proof of:
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*).

Definition 541 (*tensor product of σ-algebras*)
is explicitly cited in the proof of:
Lemma 542 (*product of measurable subsets is measurable*),
Lemma 543 (*measurability of function to product space*),
Lemma 546 (*generating product measurable space*),
Lemma 551 (*measurability of section*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 542 (*product of measurable subsets is measurable*)
is explicitly cited in the proof of:
Lemma 543 (*measurability of function to product space*),
Lemma 825 (*measure of section of product*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 833 (*tensor product of σ-finite measures*),
Lemma 835 (*uniqueness of tensor product measure (finite)*),
Lemma 837 (*uniqueness of tensor product measure*).

Lemma 543 (*measurability of function to product space*)
is explicitly cited in the proof of:
Lemma 544 (*canonical projection is measurable*),
Lemma 545 (*permutation is measurable*),
Lemma 572 (*$\mathcal{M}_\mathbb{R}$ is algebra*).

Lemma 544 (*canonical projection is measurable*)
is explicitly cited in the proof of:
Lemma 545 (*permutation is measurable*),
Lemma 546 (*generating product measurable space*),
Lemma 605 (*measurability of tensor product of numeric functions*).

Lemma 545 (*permutation is measurable*)
is not yet used.

Lemma 546 (*generating product measurable space*)
is explicitly cited in the proof of:
Lemma 565 (*Borel σ-algebra of $\mathbb{R}^n$*).

Definition 548 (*section in Cartesian product*)
is explicitly cited in the proof of:
Lemma 549 (*section of product*),
Lemma 550 (*compatibility of section with set operations*),
Lemma 554 (*indicator of section*).

Lemma 549 (*section of product*)
is explicitly cited in the proof of:
Lemma 551 (*measurability of section*),
Lemma 825 (*measure of section of product*).

Lemma 550 (*compatibility of section with set operations*)
is explicitly cited in the proof of:
Lemma 551 (*measurability of section*),
Lemma 552 (*countable union of sections is measurable*),
Lemma 553 (*countable intersection of sections is measurable*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*).

Lemma 551 (*measurability of section*)
is explicitly cited in the proof of:
Lemma 552 (*countable union of sections is measurable*),
Lemma 553 (*countable intersection of sections is measurable*),
Lemma 555 (*measurability of function from product space*),
Lemma 824 (*measure of section*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 828 (*measurability of measure of section*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),

Lemma 838 (*negligibility of measurable section*),
Theorem 846 (*Tonelli*).

Lemma 552 (*countable union of sections is measurable*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*).

Lemma 553 (*countable intersection of sections is measurable*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*).

Lemma 554 (*indicator of section*)
is explicitly cited in the proof of:
Theorem 846 (*Tonelli*),
Lemma 847 (*Tonelli over subset*).

Lemma 555 (*measurability of function from product space*)
is not yet used.

Lemma 558 (*Borel σ-algebra of* $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 559 (*countable generator of Borel σ-algebra of* $\mathbb{R}$),
Lemma 563 (*Borel σ-algebra of* $\mathbb{R}_+$),
Lemma 570 (*measurability of numeric function to* $\mathbb{R}$),
Lemma 721 ($\mathcal{B}(\mathbb{R})$ *is sub-σ-algebra of* $\mathcal{L}$),
Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$).

Lemma 559 (*countable generator of Borel σ-algebra of* $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 565 (*Borel σ-algebra of* $\mathbb{R}^n$).

Lemma 560 (*Borel σ-algebra of* $\overline{\mathbb{R}}$)
is explicitly cited in the proof of:
Lemma 564 (*Borel σ-algebra of* $\overline{\mathbb{R}}_+$),
Lemma 578 (*measurability of numeric function*).

Lemma 561 (*Borel subsets of* $\overline{\mathbb{R}}$ *and* $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 577 ($\mathcal{M}$ *and finite is* $\mathcal{M}_{\mathbb{R}}$).

Lemma 563 (*Borel σ-algebra of* $\mathbb{R}_+$)
is not yet used.

Lemma 564 (*Borel σ-algebra of* $\overline{\mathbb{R}}_+$)
is not yet used.

Lemma 565 (*Borel σ-algebra of* $\mathbb{R}^n$)
is explicitly cited in the proof of:
Lemma 572 ($\mathcal{M}_{\mathbb{R}}$ *is algebra*).

Definition 567 ($\mathcal{M}_{\mathbb{R}}$, *vector space of measurable numeric functions to* $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 569 (*measurability of indicator function*),
Lemma 570 (*measurability of numeric function to* $\mathbb{R}$),
Lemma 571 (*inverse image is measurable in* $\mathbb{R}$),
Lemma 572 ($\mathcal{M}_{\mathbb{R}}$ *is algebra*),
Lemma 577 ($\mathcal{M}$ *and finite is* $\mathcal{M}_{\mathbb{R}}$),
Lemma 596 ($\mathcal{M}$ *is closed under absolute value*).

Lemma 569 (***measurability of indicator function***)
is explicitly cited in the proof of:
Lemma 591 (*measurability and masking*),
Lemma 682 (*almost sum*),
Lemma 734 (*$\mathcal{IF}$ is measurable*),
Lemma 759 (*$\mathcal{SF}$ is measurable*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*).

Lemma 570 (***measurability of numeric function to*** $\mathbb{R}$)
is not yet used.

Lemma 571 (***inverse image is measurable in*** $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 572 (***$\mathcal{M}_\mathbb{R}$ is algebra***)
is explicitly cited in the proof of:
Lemma 574 (*$\mathcal{M}_\mathbb{R}$ is vector space*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 759 (*$\mathcal{SF}$ is measurable*).

Lemma 574 (***$\mathcal{M}_\mathbb{R}$ is vector space***)
is explicitly cited in the proof of:
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

Definition 575 (***$\mathcal{M}$, set of measurable numeric functions***)
is explicitly cited in the proof of:
Lemma 577 (*$\mathcal{M}$ and finite is $\mathcal{M}_\mathbb{R}$*),
Lemma 578 (*measurability of numeric function*),
Lemma 579 (*inverse image is measurable*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 592 (*measurability of restriction*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 605 (*measurability of tensor product of numeric functions*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 577 (***$\mathcal{M}$ and finite is $\mathcal{M}_\mathbb{R}$***)
is explicitly cited in the proof of:
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 591 (*measurability and masking*),
Lemma 734 (*$\mathcal{IF}$ is measurable*),
Lemma 759 (*$\mathcal{SF}$ is measurable*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 578 (***measurability of numeric function***)
is explicitly cited in the proof of:
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),

Lemma 586 (*$\mathcal{M}$ is closed under infimum*),
Lemma 587 (*$\mathcal{M}$ is closed under supremum*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 810 (*Bienaymé–Chebyshev inequality*).

Lemma 579 (*inverse image is measurable*)
is explicitly cited in the proof of:
Lemma 580 (*$\mathcal{M}$ is closed under finite part*),
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 680 (*measurability of summability domain*).

Lemma 580 (*$\mathcal{M}$ is closed under finite part*)
is explicitly cited in the proof of:
Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*),
Lemma 583 (*$\mathcal{M}$ is closed under multiplication*),
Lemma 595 (*$\mathcal{M}_+$ is closed under finite part*).

Lemma 581 (*$\mathcal{M}$ is closed under addition when defined*)
is explicitly cited in the proof of:
Lemma 582 (*$\mathcal{M}$ is closed under finite sum when defined*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 597 (*$\mathcal{M}_+$ is closed under addition*),
Lemma 682 (*almost sum*),
Lemma 683 (*compatibility of almost sum with almost equality*),
Theorem 796 (*Beppo Levi, monotone convergence*).

Lemma 582 (*$\mathcal{M}$ is closed under finite sum when defined*)
is not yet used.

Lemma 583 (*$\mathcal{M}$ is closed under multiplication*)
is explicitly cited in the proof of:
Lemma 584 (*$\mathcal{M}$ is closed under finite product*),
Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*),
Lemma 591 (*measurability and masking*),
Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*),
Lemma 682 (*almost sum*).

Lemma 584 (*$\mathcal{M}$ is closed under finite product*)
is explicitly cited in the proof of:
Lemma 605 (*measurability of tensor product of numeric functions*).

Lemma 585 (*$\mathcal{M}$ is closed under scalar multiplication*)
is explicitly cited in the proof of:
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 880 (*integral is homogeneous*).

Lemma 586 (*$\mathcal{M}$ is closed under infimum*)
is explicitly cited in the proof of:
Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*),
Lemma 589 (*$\mathcal{M}$ is closed under limit superior*),
Lemma 600 (*$\mathcal{M}_+$ is closed under infimum*),
Theorem 817 (*Fatou's lemma*).

Lemma 587 (***$\mathcal{M}$ is closed under supremum***)
is explicitly cited in the proof of:
Lemma 588 (*$\mathcal{M}$ is closed under limit inferior*),
Lemma 589 (*$\mathcal{M}$ is closed under limit superior*),
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 601 (*$\mathcal{M}_+$ is closed under supremum*).

Lemma 588 (***$\mathcal{M}$ is closed under limit inferior***)
is explicitly cited in the proof of:
Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*),
Theorem 817 (*Fatou's lemma*).

Lemma 589 (***$\mathcal{M}$ is closed under limit superior***)
is explicitly cited in the proof of:
Lemma 590 (*$\mathcal{M}$ is closed under limit when pointwise convergent*).

Lemma 590 (***$\mathcal{M}$ is closed under limit when pointwise convergent***)
is explicitly cited in the proof of:
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 591 (***measurability and masking***)
is explicitly cited in the proof of:
Lemma 688 (*finite nonnegative part*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 592 (***measurability of restriction***)
is explicitly cited in the proof of:
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

Definition 593 (***$\mathcal{M}_+$, subset of nonnegative measurable numeric functions***)
is explicitly cited in the proof of:
Lemma 594 (*measurability of nonnegative and nonpositive parts*),
Lemma 595 (*$\mathcal{M}_+$ is closed under finite part*),
Lemma 596 (*$\mathcal{M}$ is closed under absolute value*),
Lemma 597 (*$\mathcal{M}_+$ is closed under addition*),
Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*),
Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*),
Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*),
Lemma 734 (*$\mathcal{IF}$ is measurable*),
Lemma 769 (*$\mathcal{SF}_+$ is measurable*),
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 848 (*Tonelli for tensor product*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 594 (***measurability of nonnegative and nonpositive parts***)
is explicitly cited in the proof of:
Lemma 688 (*finite nonnegative part*),
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Lemma 852 (*integrable is measurable*),
Lemma 853 (*equivalent definition of integrability*),

Lemma 880 (*integral is homogeneous*),
Lemma 883 (*integral is additive*).

Lemma 595 (*$\mathcal{M}_+$ is closed under finite part*)
is explicitly cited in the proof of:
Lemma 688 (*finite nonnegative part*).

Lemma 596 (*$\mathcal{M}$ is closed under absolute value*)
is explicitly cited in the proof of:
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 856 (*almost bounded by integrable is integrable*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*),
Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*).

Lemma 597 (*$\mathcal{M}_+$ is closed under addition*)
is explicitly cited in the proof of:
Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*),
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*),
Theorem 846 (*Tonelli*).

Lemma 598 (*$\mathcal{M}_+$ is closed under multiplication*)
is explicitly cited in the proof of:
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*).

Lemma 599 (*$\mathcal{M}_+$ is closed under nonnegative scalar multiplication*)
is explicitly cited in the proof of:
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*),
Lemma 827 (*measurability of measure of section (finite)*),
Theorem 846 (*Tonelli*).

Lemma 600 (*$\mathcal{M}_+$ is closed under infimum*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*).

Lemma 601 (*$\mathcal{M}_+$ is closed under supremum*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 828 (*measurability of measure of section*).

Lemma 602 (*$\mathcal{M}_+$ is closed under limit when pointwise convergent*)
is explicitly cited in the proof of:
Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Theorem 817 (*Fatou's lemma*),
Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*),
Theorem 846 (*Tonelli*).

Lemma 603 (*$\mathcal{M}_+$ is closed under countable sum*)
is explicitly cited in the proof of:
Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 827 (*measurability of measure of section (finite)*).

Definition 604 (*tensor product of numeric functions*)
is explicitly cited in the proof of:
Lemma 605 (*measurability of tensor product of numeric functions*),
Lemma 848 (*Tonelli for tensor product*).

Lemma 605 (*measurability of tensor product of numeric functions*)
is explicitly cited in the proof of:
Lemma 848 (*Tonelli for tensor product*).

Definition 607 (*additivity over measurable space*)
is explicitly cited in the proof of:
Lemma 610 (*σ-additivity implies additivity*),
Lemma 621 (*equivalent definition of measure*),
Lemma 742 (*integral in $\mathcal{IF}$ is additive*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Definition 608 (*σ-additivity over measurable space*)
is explicitly cited in the proof of:
Lemma 610 (*σ-additivity implies additivity*),
Lemma 613 (*measure over countable pseudopartition*),
Lemma 614 (*measure is monotone*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 617 (*measure is continuous from below*),
Lemma 621 (*equivalent definition of measure*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Lemma 671 (*counting measure*),
Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*),
Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*).

Lemma 610 (*σ-additivity implies additivity*)
is explicitly cited in the proof of:
Lemma 621 (*equivalent definition of measure*).

Definition 611 (*measure*)
is explicitly cited in the proof of:
Lemma 613 (*measure over countable pseudopartition*),
Lemma 614 (*measure is monotone*),
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 617 (*measure is continuous from below*),
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 621 (*equivalent definition of measure*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 628 (*trace measure*),
Lemma 629 (*restricted measure*),
Lemma 634 (*equivalent definition of considerable subset*),
Lemma 637 (*empty set is negligible*),
Lemma 638 (*compatibility of null measure with countable union*),
Lemma 668 (*uniqueness of measures extended from a π-system*),

Lemma 669 (*trivial measure*),
Lemma 670 (*equivalent definition of trivial measure*),
Lemma 671 (*counting measure*),
Lemma 673 (*σ-finite counting measure*),
Lemma 687 (*masking almost nowhere*),
Lemma 688 (*finite nonnegative part*),
Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*),
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Lemma 824 (*measure of section*),
Lemma 825 (*measure of section of product*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 832 (*tensor product of finite measures*),
Lemma 893 (*constant function is $\mathcal{L}^1$*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 613 (*measure over countable pseudopartition*)
is explicitly cited in the proof of:
Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*),
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*).

Lemma 614 (*measure is monotone*)
is explicitly cited in the proof of:
Lemma 615 (*measure satisfies the finite Boole inequality*),
Lemma 617 (*measure is continuous from below*),
Lemma 619 (*measure is continuous from above*),
Lemma 623 (*finite measure is bounded*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Lemma 670 (*equivalent definition of trivial measure*).

Lemma 615 (*measure satisfies the finite Boole inequality*)
is explicitly cited in the proof of:
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 638 (*compatibility of null measure with countable union*),
Lemma 688 (*finite nonnegative part*).

Definition 616 (*continuity from below*)
is explicitly cited in the proof of:
Lemma 617 (*measure is continuous from below*),
Lemma 619 (*measure is continuous from above*),
Lemma 621 (*equivalent definition of measure*),
Lemma 668 (*uniqueness of measures extended from a π-system*),
Theorem 796 (*Beppo Levi, monotone convergence*).

Lemma 617 (*measure is continuous from below*)
is explicitly cited in the proof of:
Lemma 619 (*measure is continuous from above*),
Lemma 620 (*measure satisfies the Boole inequality*),
Lemma 621 (*equivalent definition of measure*),

Lemma 668 (*uniqueness of measures extended from a π-system*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 828 (*measurability of measure of section*),
Lemma 835 (*uniqueness of tensor product measure (finite)*),
Lemma 837 (*uniqueness of tensor product measure*).

Definition 618 (*continuity from above*)
is explicitly cited in the proof of:
Lemma 619 (*measure is continuous from above*).

Lemma 619 (*measure is continuous from above*)
is explicitly cited in the proof of:
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 620 (*measure satisfies the Boole inequality*)
is explicitly cited in the proof of:
Lemma 638 (*compatibility of null measure with countable union*).

Lemma 621 (*equivalent definition of measure*)
is explicitly cited in the proof of:
Lemma 742 (*integral in $\mathcal{IF}$ is additive*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Definition 622 (*finite measure*)
is explicitly cited in the proof of:
Lemma 627 (*finite measure is σ-finite*),
Lemma 672 (*finiteness of counting measure*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 828 (*measurability of measure of section*),
Lemma 832 (*tensor product of finite measures*),
Lemma 837 (*uniqueness of tensor product measure*).

Lemma 623 (*finite measure is bounded*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Definition 624 (*σ-finite measure*)
is explicitly cited in the proof of:
Lemma 625 (*equivalent definition of σ-finite measure*),
Lemma 627 (*finite measure is σ-finite*),
Lemma 673 (*σ-finite counting measure*),
Lemma 728 (*Lebesgue measure is σ-finite*),
Lemma 833 (*tensor product of σ-finite measures*).

Lemma 625 (*equivalent definition of σ-finite measure*)
is explicitly cited in the proof of:
Lemma 673 (*σ-finite counting measure*),
Lemma 828 (*measurability of measure of section*),
Lemma 833 (*tensor product of σ-finite measures*),
Lemma 837 (*uniqueness of tensor product measure*).

Definition 626 (*diffuse measure*)
is explicitly cited in the proof of:

Lemma 729 (*Lebesgue measure is diffuse*),
Lemma 843 (*Lebesgue measure on* $\mathbb{R}^2$ *is diffuse*),
Lemma 867 (*integral over interval*).

Lemma 627 (*finite measure is* σ*-finite*)
is explicitly cited in the proof of:
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 628 (*trace measure*)
is explicitly cited in the proof of:
Lemma 743 (*integral in* $\mathcal{IF}$ *over subset*).

Lemma 629 (*restricted measure*)
is explicitly cited in the proof of:
Lemma 828 (*measurability of measure of section*),
Lemma 837 (*uniqueness of tensor product measure*).

Definition 631 (*negligible subset*)
is explicitly cited in the proof of:
Lemma 634 (*equivalent definition of considerable subset*),
Lemma 636 (*negligibility of measurable subset*),
Lemma 639 (**N** *is closed under countable union*),
Lemma 640 (*subset of negligible is negligible*),
Lemma 687 (*masking almost nowhere*),
Lemma 807 (*compatibility of integral in* $\mathcal{M}_+$ *with almost binary relation*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Definition 632 (*complete measure*)
is not yet used.

Definition 633 (*considerable subset*)
is explicitly cited in the proof of:
Lemma 634 (*equivalent definition of considerable subset*).

Lemma 634 (*equivalent definition of considerable subset*)
is not yet used.

Lemma 636 (*negligibility of measurable subset*)
is explicitly cited in the proof of:
Lemma 637 (*empty set is negligible*),
Lemma 688 (*finite nonnegative part*),
Lemma 806 (*integral in* $\mathcal{M}_+$ *is almost definite*),
Lemma 807 (*compatibility of integral in* $\mathcal{M}_+$ *with almost binary relation*),
Lemma 811 (*integrable in* $\mathcal{M}_+$ *is almost finite*),
Lemma 838 (*negligibility of measurable section*).

Lemma 637 (*empty set is negligible*)
is explicitly cited in the proof of:
Lemma 643 (*everywhere implies almost everywhere*),
Lemma 683 (*compatibility of almost sum with almost equality*).

Lemma 638 (*compatibility of null measure with countable union*)
is explicitly cited in the proof of:
Lemma 639 (**N** *is closed under countable union*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 639 (*N is closed under countable union*)
is explicitly cited in the proof of:
Lemma 646 (*extended almost modus ponens*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*),
Lemma 659 (*compatibility of almost binary relation with operator*).

Lemma 640 (*subset of negligible is negligible*)
is explicitly cited in the proof of:
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Lemma 646 (*extended almost modus ponens*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*).

Definition 641 (*property almost satisfied*)
is explicitly cited in the proof of:
Lemma 643 (*everywhere implies almost everywhere*),
Lemma 644 (*everywhere implies almost everywhere for almost the same*),
Lemma 646 (*extended almost modus ponens*),
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*),
Lemma 659 (*compatibility of almost binary relation with operator*),
Lemma 681 (*negligibility of summability domain*),
Lemma 682 (*almost sum*),
Lemma 687 (*masking almost nowhere*),
Lemma 688 (*finite nonnegative part*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 643 (*everywhere implies almost everywhere*)
is explicitly cited in the proof of:
Lemma 647 (*almost modus ponens*),
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 857 (*bounded by integrable is integrable*),
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

Lemma 644 (*everywhere implies almost everywhere for almost the same*)
is explicitly cited in the proof of:
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*),
Lemma 659 (*compatibility of almost binary relation with operator*),
Lemma 664 (*definiteness implies almost definiteness*).

Lemma 646 (*extended almost modus ponens*)
is explicitly cited in the proof of:
Lemma 647 (*almost modus ponens*).

Lemma 647 (*almost modus ponens*)
is explicitly cited in the proof of:
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 664 (*definiteness implies almost definiteness*).

Definition 649 (*almost definition*)
is explicitly cited in the proof of:
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*),
Lemma 659 (*compatibility of almost binary relation with operator*).

Definition 650 (*almost binary relation*)
is explicitly cited in the proof of:
Lemma 651 (*compatibility of almost binary relation with reflexivity*),
Lemma 652 (*compatibility of almost binary relation with symmetry*),
Lemma 653 (*compatibility of almost binary relation with antisymmetry*),
Lemma 654 (*compatibility of almost binary relation with transitivity*),
Lemma 659 (*compatibility of almost binary relation with operator*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

Lemma 651 (*compatibility of almost binary relation with reflexivity*)
is explicitly cited in the proof of:
Lemma 656 (*almost equivalence is equivalence relation*),
Lemma 658 (*almost order is order relation*).

Lemma 652 (*compatibility of almost binary relation with symmetry*)
is explicitly cited in the proof of:
Lemma 656 (*almost equivalence is equivalence relation*).

Lemma 653 (*compatibility of almost binary relation with antisymmetry*)
is explicitly cited in the proof of:
Lemma 658 (*almost order is order relation*).

Lemma 654 (*compatibility of almost binary relation with transitivity*)
is explicitly cited in the proof of:
Lemma 656 (*almost equivalence is equivalence relation*),
Lemma 658 (*almost order is order relation*).

Lemma 656 (*almost equivalence is equivalence relation*)
is explicitly cited in the proof of:
Lemma 657 (*almost equality is equivalence relation*).

Lemma 657 (*almost equality is equivalence relation*)
is explicitly cited in the proof of:
Lemma 683 (*compatibility of almost sum with almost equality*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*).

Lemma 658 (*almost order is order relation*)
is explicitly cited in the proof of:
Lemma 856 (*almost bounded by integrable is integrable*).

Lemma 659 (*compatibility of almost binary relation with operator*)
is explicitly cited in the proof of:
Lemma 660 (*compatibility of almost equality with operator*),
Lemma 661 (*compatibility of almost inequality with operator*).

Lemma 660 (*compatibility of almost equality with operator*)
is explicitly cited in the proof of:
Lemma 683 (*compatibility of almost sum with almost equality*),
Lemma 686 (*absolute value is almost definite*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 877 (*compatibility of N_1 with almost equality*),
Lemma 894 (*first mean value theorem*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 661 (*compatibility of almost inequality with operator*)
is not yet used.

Lemma 664 (*definiteness implies almost definiteness*)
is explicitly cited in the proof of:
Lemma 686 (*absolute value is almost definite*).

Lemma 668 (*uniqueness of measures extended from a π-system*)
is explicitly cited in the proof of:
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*).

Lemma 669 (*trivial measure*)
is explicitly cited in the proof of:
Lemma 670 (*equivalent definition of trivial measure*).

Lemma 670 (*equivalent definition of trivial measure*)
is not yet used.

Lemma 671 (*counting measure*)
is explicitly cited in the proof of:
Lemma 672 (*finiteness of counting measure*),
Lemma 673 (*σ-finite counting measure*),
Lemma 676 (*equivalent definition of Dirac measure*),
Lemma 677 (*Dirac measure is finite*),
Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*).

Lemma 672 (*finiteness of counting measure*)
is explicitly cited in the proof of:
Lemma 677 (*Dirac measure is finite*).

Lemma 673 (*σ-finite counting measure*)
is not yet used.

Definition 675 (*Dirac measure*)
is explicitly cited in the proof of:
Lemma 676 (*equivalent definition of Dirac measure*),
Lemma 677 (*Dirac measure is finite*),
Lemma 787 (*integral in $\mathcal{SF}_+$ for Dirac measure*),
Lemma 822 (*integral in $\mathcal{M}_+$ for Dirac measure*),
Lemma 872 (*integral for Dirac measure*).

Lemma 676 (*equivalent definition of Dirac measure*)
is not yet used.

Lemma 677 (*Dirac measure is finite*)
is not yet used.

Definition 678 (*summability domain*)
is explicitly cited in the proof of:
Lemma 679 (*summability on summability domain*),
Lemma 685 (*almost sum is sum*),
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*).

Lemma 679 (*summability on summability domain*)
is explicitly cited in the proof of:
Lemma 681 (*negligibility of summability domain*).

Lemma 680 (*measurability of summability domain*)
is explicitly cited in the proof of:
Lemma 682 (*almost sum*).

Lemma 681 (*negligibility of summability domain*)
is explicitly cited in the proof of:
Lemma 682 (*almost sum*),
Lemma 683 (*compatibility of almost sum with almost equality*).

Lemma 682 (*almost sum*)
is explicitly cited in the proof of:
Lemma 683 (*compatibility of almost sum with almost equality*),
Lemma 685 (*almost sum is sum*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Lemma 683 (*compatibility of almost sum with almost equality*)
is explicitly cited in the proof of:
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Lemma 685 (*almost sum is sum*)
is explicitly cited in the proof of:
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*).

Lemma 686 (*absolute value is almost definite*)
is explicitly cited in the proof of:
Lemma 878 (*N_1 is almost definite*).

Lemma 687 (*masking almost nowhere*)
is explicitly cited in the proof of:
Lemma 688 (*finite nonnegative part*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 688 (*finite nonnegative part*)
is explicitly cited in the proof of:
Theorem 899 (*Lebesgue, extended dominated convergence*).

Definition 691 (*length of interval*)
is explicitly cited in the proof of:
Lemma 692 (*length is nonnegative*),
Lemma 693 (*length is homogeneous*),
Lemma 694 (*length of partition*),
Lemma 700 (*$\lambda^\star$ is homogeneous*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*).

Lemma 692 (*length is nonnegative*)
is explicitly cited in the proof of:
Lemma 699 (*$\lambda^\star$ is nonnegative*).

Lemma 693 (*length is homogeneous*)
is explicitly cited in the proof of:
Lemma 694 (*length of partition*).

Lemma 694 (*length of partition*)
is not yet used.

Definition 695 (*set of countable cover with bounded open intervals*)
is explicitly cited in the proof of:
Lemma 696 (*set of countable cover with bounded open intervals is nonempty*),
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*).

Lemma 696 (*set of countable cover with bounded open intervals is nonempty*)
is not yet used.

Definition 697 (*$\lambda^\star$, Lebesgue measure candidate*)
is explicitly cited in the proof of:
Lemma 700 (*$\lambda^\star$ is homogeneous*),
Lemma 701 (*$\lambda^\star$ is monotone*),
Lemma 702 (*$\lambda^\star$ is σ-subadditive*),
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 717 (*rays are Lebesgue-measurable*).

Lemma 699 (*$\lambda^\star$ is nonnegative*)
is explicitly cited in the proof of:
Lemma 700 (*$\lambda^\star$ is homogeneous*),
Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*).

Lemma 700 (*$\lambda^\star$ is homogeneous*)
is explicitly cited in the proof of:
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 711 (*$\mathcal{L}$ is set algebra*),
Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*).

Lemma 701 (*$\lambda^\star$ is monotone*)
is explicitly cited in the proof of:
Lemma 703 (*$\lambda^\star$ generalizes length of interval*),
Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*),
Lemma 717 (*rays are Lebesgue-measurable*).

Lemma 702 (*$\lambda^\star$ is σ-subadditive*)
is explicitly cited in the proof of:
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 709 (*$\mathcal{L}$ is closed under finite union*),
Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*),
Lemma 717 (*rays are Lebesgue-measurable*).

Lemma 703 (*$\lambda^\star$ generalizes length of interval*)
is explicitly cited in the proof of:
Lemma 717 (*rays are Lebesgue-measurable*),
Theorem 724 (*Carathéodory, Lebesgue measure on $\mathbb{R}$*),
Lemma 726 (*Lebesgue measure generalizes length of interval*),
Lemma 728 (*Lebesgue measure is σ-finite*),
Lemma 729 (*Lebesgue measure is diffuse*).

Definition 705 (*$\mathcal{L}$, Lebesgue σ-algebra*)
is explicitly cited in the proof of:
Lemma 707 (*equivalent definition of $\mathcal{L}$*),
Lemma 708 (*$\mathcal{L}$ is closed under complement*),
Lemma 709 (*$\mathcal{L}$ is closed under finite union*),
Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

Lemma 707 (*equivalent definition of $\mathcal{L}$*)
is explicitly cited in the proof of:
Lemma 709 (*$\mathcal{L}$ is closed under finite union*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*),
Lemma 717 (*rays are Lebesgue-measurable*).

Lemma 708 (*$\mathcal{L}$ is closed under complement*)
is explicitly cited in the proof of:
Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*),
Lemma 711 (*$\mathcal{L}$ is set algebra*),
Lemma 717 (*rays are Lebesgue-measurable*).

Lemma 709 (*$\mathcal{L}$ is closed under finite union*)
is explicitly cited in the proof of:
Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

Lemma 710 (*$\mathcal{L}$ is closed under finite intersection*)
is explicitly cited in the proof of:
Lemma 711 (*$\mathcal{L}$ is set algebra*),
Lemma 718 (*intervals are Lebesgue-measurable*).

Lemma 711 (*$\mathcal{L}$ is set algebra*)
is explicitly cited in the proof of:
Lemma 715 (*partition of countable union in $\mathcal{L}$*),
Lemma 719 (*$\mathcal{L}$ is σ-algebra*).

Lemma 712 (*$\lambda^\star$ is additive on $\mathcal{L}$*)
is explicitly cited in the proof of:
Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*),
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

Lemma 714 (*$\lambda^\star$ is σ-additive on $\mathcal{L}$*)
is explicitly cited in the proof of:
Lemma 720 (*$\lambda^\star$ is measure on $\mathcal{L}$*).

Lemma 715 (*partition of countable union in $\mathcal{L}$*)
is explicitly cited in the proof of:
Lemma 716 (*$\mathcal{L}$ is closed under countable union*).

Lemma 716 (*$\mathcal{L}$ is closed under countable union*)
is explicitly cited in the proof of:
Lemma 717 (*rays are Lebesgue-measurable*),
Lemma 719 (*$\mathcal{L}$ is σ-algebra*).

Lemma 717 (*rays are Lebesgue-measurable*)
is explicitly cited in the proof of:
Lemma 718 (*intervals are Lebesgue-measurable*).

Lemma 718 (*intervals are Lebesgue-measurable*)
is explicitly cited in the proof of:
Lemma 721 ($\mathcal{B}(\mathbb{R})$ *is sub-σ-algebra of* $\mathcal{L}$).

Lemma 719 ($\mathcal{L}$ *is σ-algebra*)
is explicitly cited in the proof of:
Lemma 720 ($\lambda^\star$ *is measure on* $\mathcal{L}$).

Lemma 720 ($\lambda^\star$ *is measure on* $\mathcal{L}$)
is explicitly cited in the proof of:
Lemma 722 ($\lambda^\star$ *is measure on* $\mathcal{B}(\mathbb{R})$).

Lemma 721 ($\mathcal{B}(\mathbb{R})$ *is sub-σ-algebra of* $\mathcal{L}$)
is explicitly cited in the proof of:
Lemma 722 ($\lambda^\star$ *is measure on* $\mathcal{B}(\mathbb{R})$).

Lemma 722 ($\lambda^\star$ *is measure on* $\mathcal{B}(\mathbb{R})$)
is explicitly cited in the proof of:
Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$).

Theorem 724 (*Carathéodory, Lebesgue measure on* $\mathbb{R}$)
is explicitly cited in the proof of:
Lemma 726 (*Lebesgue measure generalizes length of interval*),
Lemma 728 (*Lebesgue measure is σ-finite*),
Lemma 729 (*Lebesgue measure is diffuse*),
Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$).

Lemma 726 (*Lebesgue measure generalizes length of interval*)
is explicitly cited in the proof of:
Lemma 840 (*Lebesgue measure on* $\mathbb{R}^2$ *generalizes area of boxes*).

Lemma 728 (*Lebesgue measure is σ-finite*)
is explicitly cited in the proof of:
Lemma 839 (*Lebesgue measure on* $\mathbb{R}^2$),
Lemma 842 (*Lebesgue measure on* $\mathbb{R}^2$ *is σ-finite*).

Lemma 729 (*Lebesgue measure is diffuse*)
is not yet used.

Definition 732 ($\mathcal{IF}$, *set of measurable indicator functions*)
is explicitly cited in the proof of:
Lemma 733 (*indicator and support are each other inverse*),
Lemma 734 ($\mathcal{IF}$ *is measurable*),
Lemma 735 ($\mathcal{IF}$ *is σ-additive*),
Lemma 736 ($\mathcal{IF}$ *is closed under multiplication*),
Lemma 738 ($\mathcal{IF}$ *is closed under extension by zero*),
Lemma 739 ($\mathcal{IF}$ *is closed under restriction*),
Lemma 749 ($\mathcal{SF}$ *simple representation*),
Lemma 769 ($\mathcal{SF}_+$ *is measurable*),
Lemma 771 (*integral in* $\mathcal{SF}_+$ *generalizes integral in* $\mathcal{IF}$),
Lemma 784 (*integral in* $\mathcal{SF}_+$ *over subset is additive*).

Lemma 733 (*indicator and support are each other inverse*)
is explicitly cited in the proof of:
Lemma 741 (*equivalent definition of integral in* $\mathcal{IF}$),
Lemma 743 (*integral in* $\mathcal{IF}$ *over subset*),
Lemma 744 (*integral in* $\mathcal{IF}$ *over subset is additive*),

Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*),
Lemma 749 (*$\mathcal{SF}$ simple representation*),
Theorem 846 (*Tonelli*).

Lemma 734 (*$\mathcal{IF}$ is measurable*)
is not yet used.

Lemma 735 (*$\mathcal{IF}$ is σ-additive*)
is explicitly cited in the proof of:
Lemma 742 (*integral in $\mathcal{IF}$ is additive*),
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*).

Lemma 736 (*$\mathcal{IF}$ is closed under multiplication*)
is explicitly cited in the proof of:
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*),
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*).

Lemma 738 (*$\mathcal{IF}$ is closed under extension by zero*)
is explicitly cited in the proof of:
Lemma 743 (*integral in $\mathcal{IF}$ over subset*),
Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*).

Lemma 739 (*$\mathcal{IF}$ is closed under restriction*)
is explicitly cited in the proof of:
Lemma 743 (*integral in $\mathcal{IF}$ over subset*),
Lemma 761 (*$\mathcal{SF}$ is closed under restriction*).

Definition 740 (*integral in $\mathcal{IF}$*)
is explicitly cited in the proof of:
Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*).

Lemma 741 (*equivalent definition of integral in $\mathcal{IF}$*)
is explicitly cited in the proof of:
Lemma 742 (*integral in $\mathcal{IF}$ is additive*),
Lemma 743 (*integral in $\mathcal{IF}$ over subset*),
Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*),
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*).

Lemma 742 (*integral in $\mathcal{IF}$ is additive*)
is explicitly cited in the proof of:
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*).

Lemma 743 (*integral in $\mathcal{IF}$ over subset*)
is explicitly cited in the proof of:
Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*).

Lemma 744 (*integral in $\mathcal{IF}$ over subset is additive*)
is not yet used.

Lemma 746 (*integral in $\mathcal{IF}$ for counting measure*)
is explicitly cited in the proof of:
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*).

Definition 748 (*$\mathcal{SF}$, vector space of simple functions*)
is explicitly cited in the proof of:
Lemma 749 (*$\mathcal{SF}$ simple representation*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 759 (*$\mathcal{SF}$ is measurable*),
Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*),
Lemma 761 (*$\mathcal{SF}$ is closed under restriction*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*),
Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 893 (*constant function is $\mathcal{L}^1$*).

Lemma 749 (*$\mathcal{SF}$ simple representation*)
is explicitly cited in the proof of:
Lemma 752 (*$\mathcal{SF}$ canonical representation*),
Lemma 754 (*$\mathcal{SF}$ disjoint representation*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 767 (*$\mathcal{SF}_+$ simple representation*).

Lemma 752 (*$\mathcal{SF}$ canonical representation*)
is explicitly cited in the proof of:
Lemma 754 (*$\mathcal{SF}$ disjoint representation*),
Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*),
Lemma 765 (*$\mathcal{SF}_+$ canonical representation*),
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*).

Lemma 754 (*$\mathcal{SF}$ disjoint representation*)
is explicitly cited in the proof of:
Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*),
Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*),
Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*).

Lemma 756 (*$\mathcal{SF}$ disjoint representation is subpartition of canonical representation*)
is explicitly cited in the proof of:
Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*).

Lemma 757 (*$\mathcal{SF}$ is algebra over $\mathbb{R}$*)
is explicitly cited in the proof of:
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*).

Lemma 759 (*$\mathcal{SF}$ is measurable*)
is explicitly cited in the proof of:
Lemma 769 (*$\mathcal{SF}_+$ is measurable*),
Theorem 796 (*Beppo Levi, monotone convergence*).

Lemma 760 (*$\mathcal{SF}$ is closed under extension by zero*)
is explicitly cited in the proof of:
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*).

Lemma 761 (*$\mathcal{SF}$ is closed under restriction*)
is explicitly cited in the proof of:
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

Definition 763 (*$\mathcal{SF}_+$, subset of nonnegative simple functions*)
is explicitly cited in the proof of:
Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*),
Lemma 765 (*$\mathcal{SF}_+$ canonical representation*),
Lemma 767 (*$\mathcal{SF}_+$ simple representation*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 769 (*$\mathcal{SF}_+$ is measurable*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*),
Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*).

Lemma 764 (*$\mathcal{SF}_+$ disjoint representation*)
is explicitly cited in the proof of:
Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*),
Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*),
Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*).

Lemma 765 (*$\mathcal{SF}_+$ canonical representation*)
is explicitly cited in the proof of:
Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*),
Lemma 767 (*$\mathcal{SF}_+$ simple representation*),
Lemma 770 (*integral in $\mathcal{SF}_+$*),
Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*).

Lemma 766 (*$\mathcal{SF}_+$ disjoint representation is subpartition of canonical representation*)

is explicitly cited in the proof of:
Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*).

Lemma 767 (*$\mathcal{SF}_+$ simple representation*)
is explicitly cited in the proof of:
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*),
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*),
Theorem 846 (*Tonelli*).

Lemma 768 (*$\mathcal{SF}_+$ is closed under positive algebra operations*)
is explicitly cited in the proof of:
Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*).

Lemma 769 (*$\mathcal{SF}_+$ is measurable*)
is explicitly cited in the proof of:
Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*),
Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*),
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*).

Lemma 770 (*integral in $\mathcal{SF}_+$*)
is explicitly cited in the proof of:
Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*),
Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*),

Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*),
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*),
Lemma 789 (*integral in $\mathcal{M}_+$*),
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 893 (*constant function is $\mathcal{L}^1$*).

Lemma 771 (*integral in $\mathcal{SF}_+$ generalizes integral in $\mathcal{IF}$*)
is explicitly cited in the proof of:
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*),
Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*),
Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*),
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*).

Lemma 772 (*equivalent definition of the integral in $\mathcal{SF}_+$ (disjoint)*)
is explicitly cited in the proof of:
Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*).

Lemma 774 (*integral in $\mathcal{SF}_+$ is additive*)
is explicitly cited in the proof of:
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*).

Lemma 775 (*decomposition of measure in $\mathcal{SF}_+$*)
is explicitly cited in the proof of:
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 776 (*change of variable in sum in $\mathcal{SF}_+$*)
is explicitly cited in the proof of:
Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*).

Lemma 778 (*integral in $\mathcal{SF}_+$ is additive (alternate proof)*)
is explicitly cited in the proof of:
Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*).

Lemma 779 (*integral in $\mathcal{SF}_+$ is positive linear*)
is explicitly cited in the proof of:
Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*),
Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*),
Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*).

Lemma 780 (*equivalent definition of the integral in $\mathcal{SF}_+$ (simple)*)
is explicitly cited in the proof of:
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Theorem 796 (*Beppo Levi, monotone convergence*).

Lemma 781 (*integral in $\mathcal{SF}_+$ is monotone*)
is explicitly cited in the proof of:
Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*).

Lemma 782 (*integral in $\mathcal{SF}_+$ is continuous*)
is explicitly cited in the proof of:
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*).

Lemma 783 (*integral in $\mathcal{SF}_+$ over subset*)
is explicitly cited in the proof of:
Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

Lemma 784 (*integral in $\mathcal{SF}_+$ over subset is additive*)
is not yet used.

Lemma 785 (*integral in $\mathcal{SF}_+$ for counting measure*)
is explicitly cited in the proof of:
Lemma 786 (*integral in $\mathcal{SF}_+$ for counting measure on $\mathbb{N}$*),
Lemma 787 (*integral in $\mathcal{SF}_+$ for Dirac measure*),
Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*).

Lemma 786 (*integral in $\mathcal{SF}_+$ for counting measure on $\mathbb{N}$*)
is not yet used.

Lemma 787 (*integral in $\mathcal{SF}_+$ for Dirac measure*)
is not yet used.

Lemma 789 (*integral in $\mathcal{M}_+$*)
is explicitly cited in the proof of:
Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*),
Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*),
Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*),
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*),
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*),
Lemma 812 (*bounded by integrable in $\mathcal{M}_+$ is integrable*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 852 (*integrable is measurable*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 856 (*almost bounded by integrable is integrable*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 864 (*integral over subset*),
Lemma 866 (*integral over singleton*),
Lemma 869 (*integral for counting measure*),
Lemma 874 (*seminorm $\mathcal{L}^1$*),
Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*),
Lemma 880 (*integral is homogeneous*),
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 790 (*integral in $\mathcal{M}_+$ generalizes integral in $\mathcal{SF}_+$*)
is explicitly cited in the proof of:
Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*),
Lemma 800 (*usage of adapted sequences*),
Lemma 893 (*constant function is $\mathcal{L}^1$*).

Lemma 791 (*integral in $\mathcal{M}_+$ of indicator function*)
is explicitly cited in the proof of:
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*),

Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Theorem 846 (*Tonelli*).

Lemma 792 (*integral in $\mathcal{M}_+$ is positive homogeneous*)
is explicitly cited in the proof of:
Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*),
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*),
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 848 (*Tonelli for tensor product*),
Lemma 879 (*N_1 is absolutely homogeneous*),
Lemma 880 (*integral is homogeneous*).

Lemma 793 (*integral in $\mathcal{M}_+$ of zero is zero*)
is explicitly cited in the proof of:
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 860 (*integral of zero is zero*).

Lemma 794 (*integral in $\mathcal{M}_+$ is monotone*)
is explicitly cited in the proof of:
Theorem 796 (*Beppo Levi, monotone convergence*),
Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 812 (*bounded by integrable in $\mathcal{M}_+$ is integrable*),
Theorem 817 (*Fatou's lemma*),
Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*),
Lemma 865 (*integral over subset is σ-additive*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Theorem 796 (*Beppo Levi, monotone convergence*)
is explicitly cited in the proof of:
Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*),
Lemma 800 (*usage of adapted sequences*),
Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*),
Theorem 817 (*Fatou's lemma*).

Lemma 797 (*integral in $\mathcal{M}_+$ is homogeneous at ∞*)
is explicitly cited in the proof of:
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*),
Lemma 810 (*Bienaymé–Chebyshev inequality*),
Lemma 879 (*N_1 is absolutely homogeneous*).

Definition 798 (*adapted sequence*)
is explicitly cited in the proof of:
Lemma 799 (*adapted sequence in $\mathcal{M}_+$*),
Lemma 800 (*usage of adapted sequences*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*).

Lemma 799 (*adapted sequence in $\mathcal{M}_+$*)
is explicitly cited in the proof of:
Lemma 800 (*usage of adapted sequences*),
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*),
Theorem 846 (*Tonelli*).

Lemma 800 (*usage of adapted sequences*)
is explicitly cited in the proof of:
Lemma 801 (*integral in $\mathcal{M}_+$ is additive*),
Lemma 813 (*integral in $\mathcal{M}_+$ over subset*),
Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*),
Theorem 846 (*Tonelli*).

Lemma 801 (*integral in $\mathcal{M}_+$ is additive*)
is explicitly cited in the proof of:
Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*),
Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*),
Lemma 805 (*compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts*),
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Lemma 802 (*integral in $\mathcal{M}_+$ is positive linear*)
is explicitly cited in the proof of:
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Theorem 846 (*Tonelli*).

Lemma 803 (*integral in $\mathcal{M}_+$ is σ-additive*)
is explicitly cited in the proof of:
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 865 (*integral over subset is σ-additive*).

Lemma 804 (*integral in $\mathcal{M}_+$ of decomposition into nonpositive and nonnegative parts*)

is explicitly cited in the proof of:
Lemma 853 (*equivalent definition of integrability*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 805 (*compatibility of integral in $\mathcal{M}_+$ with nonpositive and nonnegative parts*)

is explicitly cited in the proof of:
Lemma 883 (*integral is additive*).

Lemma 806 (*integral in $\mathcal{M}_+$ is almost definite*)
is explicitly cited in the proof of:
Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*),
Lemma 838 (*negligibility of measurable section*),
Lemma 878 (*N_1 is almost definite*),
Lemma 894 (*first mean value theorem*).

Lemma 807 (*compatibility of integral in $\mathcal{M}_+$ with almost binary relation*)
is explicitly cited in the proof of:
Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*),
Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*).

Lemma 808 (*compatibility of integral in $\mathcal{M}_+$ with almost equality*)
is explicitly cited in the proof of:
Lemma 862 (*compatibility of integral with almost equality*).

Lemma 809 (*integral in $\mathcal{M}_+$ is almost monotone*)
is explicitly cited in the proof of:
Lemma 856 (*almost bounded by integrable is integrable*).

Lemma 810 (*Bienaymé–Chebyshev inequality*)
is explicitly cited in the proof of:
Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*).

Lemma 811 (*integrable in $\mathcal{M}_+$ is almost finite*)
is explicitly cited in the proof of:
Lemma 855 (*integrable is almost finite*).

Lemma 812 (*bounded by integrable in $\mathcal{M}_+$ is integrable*)
is not yet used.

Lemma 813 (*integral in $\mathcal{M}_+$ over subset*)
is explicitly cited in the proof of:
Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*),
Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*),
Lemma 847 (*Tonelli over subset*),
Lemma 864 (*integral over subset*).

Lemma 814 (*integral in $\mathcal{M}_+$ over subset is σ-additive*)
is explicitly cited in the proof of:
Lemma 865 (*integral over subset is σ-additive*).

Lemma 815 (*integral in $\mathcal{M}_+$ over singleton*)
is explicitly cited in the proof of:
Lemma 866 (*integral over singleton*).

Theorem 817 (*Fatou's lemma*)
is explicitly cited in the proof of:
Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 818 (*integral in $\mathcal{M}_+$ of pointwise convergent sequence*)
is not yet used.

Lemma 819 (*integral in $\mathcal{M}_+$ for counting measure*)
is explicitly cited in the proof of:
Lemma 820 (*integral in $\mathcal{M}_+$ for counting measure on $\mathbb{N}$*),
Lemma 822 (*integral in $\mathcal{M}_+$ for Dirac measure*),
Lemma 869 (*integral for counting measure*).

Lemma 820 (*integral in $\mathcal{M}_+$ for counting measure on $\mathbb{N}$*)
is not yet used.

Lemma 822 (*integral in $\mathcal{M}_+$ for Dirac measure*)
is not yet used.

Lemma 824 (*measure of section*)
is explicitly cited in the proof of:
Lemma 825 (*measure of section of product*),
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 828 (*measurability of measure of section*),
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 837 (*uniqueness of tensor product measure*),
Theorem 846 (*Tonelli*).

Lemma 825 (*measure of section of product*)
is explicitly cited in the proof of:
Lemma 827 (*measurability of measure of section (finite)*),
Lemma 831 (*candidate tensor product measure is tensor product measure*).

Lemma 827 (*measurability of measure of section (finite)*)
is explicitly cited in the proof of:
Lemma 828 (*measurability of measure of section*).

Lemma 828 (*measurability of measure of section*)
is explicitly cited in the proof of:
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Theorem 846 (*Tonelli*).

Definition 829 (*tensor product measure*)
is explicitly cited in the proof of:
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 832 (*tensor product of finite measures*),
Lemma 833 (*tensor product of σ-finite measures*),
Lemma 835 (*uniqueness of tensor product measure (finite)*),
Lemma 837 (*uniqueness of tensor product measure*),
Lemma 840 (*Lebesgue measure on $\mathbb{R}^2$ generalizes area of boxes*).

Definition 830 (*candidate tensor product measure*)
is explicitly cited in the proof of:
Lemma 831 (*candidate tensor product measure is tensor product measure*),
Lemma 837 (*uniqueness of tensor product measure*).

Lemma 831 (*candidate tensor product measure is tensor product measure*)
is explicitly cited in the proof of:
Lemma 835 (*uniqueness of tensor product measure (finite)*),
Lemma 837 (*uniqueness of tensor product measure*).

Lemma 832 (*tensor product of finite measures*)
is explicitly cited in the proof of:
Lemma 835 (*uniqueness of tensor product measure (finite)*).

Lemma 833 (*tensor product of σ-finite measures*)
is explicitly cited in the proof of:
Lemma 837 (*uniqueness of tensor product measure*),
Lemma 842 (*Lebesgue measure on $\mathbb{R}^2$ is σ-finite*).

Lemma 835 (*uniqueness of tensor product measure (finite)*)
is explicitly cited in the proof of:
Lemma 837 (*uniqueness of tensor product measure*).

Lemma 837 (*uniqueness of tensor product measure*)
is explicitly cited in the proof of:
Lemma 838 (*negligibility of measurable section*),
Lemma 839 (*Lebesgue measure on $\mathbb{R}^2$*),
Theorem 846 (*Tonelli*).

Lemma 838 (*negligibility of measurable section*)
is not yet used.

Lemma 839 (*Lebesgue measure on $\mathbb{R}^2$*)
is explicitly cited in the proof of:
Lemma 840 (*Lebesgue measure on $\mathbb{R}^2$ generalizes area of boxes*),
Lemma 842 (*Lebesgue measure on $\mathbb{R}^2$ is σ-finite*).

Lemma 840 (*Lebesgue measure on $\mathbb{R}^2$ generalizes area of boxes*)
is explicitly cited in the proof of:
Lemma 841 (*Lebesgue measure on $\mathbb{R}^2$ is zero on lines*),
Lemma 843 (*Lebesgue measure on $\mathbb{R}^2$ is diffuse*).

Lemma 841 (*Lebesgue measure on $\mathbb{R}^2$ is zero on lines*)
is not yet used.

Lemma 842 (*Lebesgue measure on $\mathbb{R}^2$ is σ-finite*)
is not yet used.

Lemma 843 (*Lebesgue measure on $\mathbb{R}^2$ is diffuse*)
is not yet used.

Definition 844 (*partial function of function from product space*)
is explicitly cited in the proof of:
Theorem 846 (*Tonelli*),
Lemma 847 (*Tonelli over subset*),
Lemma 848 (*Tonelli for tensor product*).

Theorem 846 (*Tonelli*)
is explicitly cited in the proof of:
Lemma 847 (*Tonelli over subset*),
Lemma 848 (*Tonelli for tensor product*).

Lemma 847 (*Tonelli over subset*)
is not yet used.

Lemma 848 (*Tonelli for tensor product*)
is not yet used.

Definition 851 (*integrability*)
is explicitly cited in the proof of:
Lemma 852 (*integrable is measurable*),
Lemma 853 (*equivalent definition of integrability*),
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 864 (*integral over subset*),
Lemma 866 (*integral over singleton*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 852 (*integrable is measurable*)
is explicitly cited in the proof of:
Lemma 853 (*equivalent definition of integrability*),
Lemma 880 (*integral is homogeneous*).

Lemma 853 (*equivalent definition of integrability*)
is explicitly cited in the proof of:
Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 855 (*integrable is almost finite*),
Lemma 856 (*almost bounded by integrable is integrable*),
Lemma 857 (*bounded by integrable is integrable*),
Lemma 865 (*integral over subset is σ-additive*),
Lemma 866 (*integral over singleton*),
Lemma 869 (*integral for counting measure*),
Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*),
Lemma 880 (*integral is homogeneous*),
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 854 (*compatibility of integrability in $\mathcal{M}$ and $\mathcal{M}_+$*)
is explicitly cited in the proof of:
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*).

Lemma 855 (*integrable is almost finite*)
is explicitly cited in the proof of:
Lemma 883 (*integral is additive*).

Lemma 856 (*almost bounded by integrable is integrable*)
is explicitly cited in the proof of:
Lemma 857 (*bounded by integrable is integrable*).

Lemma 857 (*bounded by integrable is integrable*)
is explicitly cited in the proof of:
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*).

Definition 858 (*integral*)
is explicitly cited in the proof of:
Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*),
Lemma 862 (*compatibility of integral with almost equality*),
Lemma 864 (*integral over subset*),
Lemma 865 (*integral over subset is σ-additive*),
Lemma 866 (*integral over singleton*),
Lemma 869 (*integral for counting measure*),
Lemma 880 (*integral is homogeneous*),
Lemma 883 (*integral is additive*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Lemma 893 (*constant function is $\mathcal{L}^1$*).

Lemma 859 (*compatibility of integral in $\mathcal{M}$ and $\mathcal{M}_+$*)
is explicitly cited in the proof of:
Lemma 860 (*integral of zero is zero*),
Lemma 893 (*constant function is $\mathcal{L}^1$*),
Lemma 894 (*first mean value theorem*).

Lemma 860 (*integral of zero is zero*)
is not yet used.

Definition 861 (*merge integral in $\mathcal{M}$ and $\mathcal{M}_+$*)
is explicitly cited in the proof of:
Lemma 874 (*seminorm $\mathcal{L}^1$*).

Lemma 862 (*compatibility of integral with almost equality*)
is explicitly cited in the proof of:
Lemma 877 (*compatibility of N_1 with almost equality*),
Lemma 883 (*integral is additive*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 864 (*integral over subset*)
is explicitly cited in the proof of:
Lemma 866 (*integral over singleton*).

Lemma 865 (*integral over subset is σ-additive*)
is explicitly cited in the proof of:
Lemma 867 (*integral over interval*),
Lemma 868 (*Chasles relation, integral over split intervals*).

Lemma 866 (*integral over singleton*)
is explicitly cited in the proof of:
Lemma 867 (*integral over interval*).

Lemma 867 (*integral over interval*)
is explicitly cited in the proof of:
Lemma 868 (*Chasles relation, integral over split intervals*).

Lemma 868 (*Chasles relation, integral over split intervals*)
is not yet used.

Lemma 869 (*integral for counting measure*)
is explicitly cited in the proof of:
Lemma 870 (*integral for counting measure on $\mathbb{N}$*),
Lemma 872 (*integral for Dirac measure*).

Lemma 870 (*integral for counting measure on $\mathbb{N}$*)
is not yet used.

Lemma 872 (*integral for Dirac measure*)
is not yet used.

Definition 873 (*integral for Lebesgue measure on $\mathbb{R}$*)
is not yet used.

Lemma 874 (*seminorm $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*),
Lemma 877 (*compatibility of N_1 with almost equality*),
Lemma 878 (*N_1 is almost definite*),
Lemma 879 (*N_1 is absolutely homogeneous*),
Lemma 880 (*integral is homogeneous*),
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*),
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 876 (*integrable is finite seminorm $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 883 (*integral is additive*).

Lemma 877 (*compatibility of N_1 with almost equality*)
is explicitly cited in the proof of:
Lemma 882 (*Minkowski inequality in $\mathcal{M}$*).

Lemma 878 (*N_1 is almost definite*)
is explicitly cited in the proof of:
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

Lemma 879 (*N_1 is absolutely homogeneous*)
is explicitly cited in the proof of:
Lemma 880 (*integral is homogeneous*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

Lemma 880 (*integral is homogeneous*)
is explicitly cited in the proof of:
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Lemma 882 (*Minkowski inequality in $\mathcal{M}$*)
is explicitly cited in the proof of:
Lemma 883 (*integral is additive*),
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*).

Lemma 883 (*integral is additive*)
is explicitly cited in the proof of:
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*).

Definition 884 (*$\mathcal{L}^1$, vector space of integrable functions*)
is explicitly cited in the proof of:
Lemma 886 (*equivalent definition of $\mathcal{L}^1$*),
Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*),
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*),
Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*),
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*),
Theorem 899 (*Lebesgue, extended dominated convergence*).

Lemma 886 (*equivalent definition of $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*).

Lemma 887 (*Minkowski inequality in $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*).

Lemma 888 (*$\mathcal{L}^1$ is seminormed vector space*)
is explicitly cited in the proof of:
Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*),
Lemma 894 (*first mean value theorem*),
Theorem 897 (*Lebesgue, dominated convergence*).

Definition 889 (*convergence in $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 890 (*$\mathcal{L}^1$ is closed under absolute value*)
is explicitly cited in the proof of:
Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 891 (*bounded by $\mathcal{L}^1$ is $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 894 (*first mean value theorem*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 892 (*integral is positive linear form on $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 894 (*first mean value theorem*),
Theorem 897 (*Lebesgue, dominated convergence*).

Lemma 893 (*constant function is $\mathcal{L}^1$*)
is explicitly cited in the proof of:
Lemma 894 (*first mean value theorem*).

Lemma 895 (*first mean value theorem*)

is explicitly cited in the proof of:

Lemma 895 (*variant of first mean value theorem*).

Lemma 895 (*variant of first mean value theorem*)

is not yet used.

Theorem 897 (*Lebesgue, dominated convergence*)

is explicitly cited in the proof of:

Theorem 899 (*Lebesgue, extended dominated convergence*).

Theorem 899 (*Lebesgue, extended dominated convergence*)

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